2.1 Concepts of Engineering Economics Analysis

*Engineering Economy:* is a collection of mathematical techniques which simplify economic comparisons.

*Time Value of Money:* means that money has different value tomorrow than it has today.

*Interest:* is a measure of the increase between the original sum borrowed or invested and the final amount owed or accrued. It is the growth in value with time.

\[
\text{Interest} = \text{Total amount accumulated} - \text{Original investment} \quad (2.1)
\]

On the other hand, if you had borrowed money from the bank at some time in the past:

\[
\text{Interest} = \text{Present amount owed} - \text{Original loan} \quad (2.2)
\]

In either case, there is an increase in the amount of money that was originally invested or borrowed, and the increase over the original amount is the interest.

*The original investment or loan:* is referred to as Principal.

### 2.1.1 Interest Calculations

When interest is expressed as a percentage of the original amount per unit time, the result is what is called an interest rate. The most common period or unit of time is one year.

\[
\% \text{ interest rate} = (\text{Interest accrued per unit of time} / \text{Original amount}) \times 100\% \quad (2.3)
\]

*Example 2.1:* Suppose you invested LE100,000 on May 1, and withdraw a total of EL106,000 exactly one year later. Compute the interest and the interest rate.

**Solution:** Interest = LE106,000 - LE100,000 = LE6,000
Percent interest rate = \((\text{LE}6,000 \text{ per year} / \text{LE}100,000) \times 100\% = 6\% \text{ per year}\)

2.1.2 Equivalence
When we are indifferent as to whether we have a quantity of money now or the assurance of some other sum of money in the future, or series of future sums of money, we say that the present sum of money is equivalent to the future sum of series of future sums. Equivalence is an essential factor in engineering economic analysis.

Different sums of money at different times can be equal in economic value. For example, if the interest rate is 6\% per year, a LE100 today (present time) would be equivalent to LE106 one year from today. Also, LE100 today is equivalent to LE94.34 one year ago. Therefore, LE94.34 last year, LE100 now, and LE106 one year from now are equivalent when the interest rate is 6\% per year. In this example, all calculations are made based on 6\% interest rate, so changing the interest rate would change the payments. As such, the equivalence is dependent on the interest rate.

2.1.3 Rate of Return
The rate of return is used when determining the profitability of a proposed investment or past investment.

\[
\text{Rate of return (RR)} = \frac{\text{(Total amount of money received} - \text{Original investment})}{\text{Original investment}} \times 100\%
\]

\[
= \frac{\text{(Profit} / \text{Original investment})}{\times 100\%}
\]

Interest rate is used when borrowing capital or when a fixed rate has been established.

2.2 Time Value of Money
The value of money is dependent on the time at which it is received. A sum of money on hand today is worth more than the same sum of money to be received in the future because the money on hand today can be invested to earn interest to gain more than the same money in the future. Thus, studying the present value of money (or the discounted
value) that will be received in the future is very important.

The money consequences of any alternative occur over a long period of time, a year or more. When money consequences occur in a short period of time, we simply add up the various sums of money and obtain a net result. But we cannot treat money this same way when the time span is longer. Which would you prefer, LE100 cash today or receiving LE100 a year from now? A little thought should convince you that it is desirable to receive the LE100 now, rather than a year from now. This is because, you might consider leaving the LE100 in a bank if you know it would be worth LE109 one year from now.

To facilitate the computations of interest formulas, the following notations will be used:

\[ i = \text{interest rate per interest period (usually calculated annually), it is stated as a decimal (e.g., 9\% interest is 0.09).} \]

\[ n = \text{number of interest periods.} \]

\[ P = \text{A present sum of money.} \]

\[ F = \text{A future sum of money.} \]

\[ A = \text{A series of periodic, equal amount of money. This is always paid or received at end of period.} \]

2.3 Single Payment

The Future Value of a given present value of money represents the amount, at some time in the future, that an investment made today will grow to if it is invested at a specific interest rate. For example, if you were to deposit LE100 today in a bank account to earn an interest rate of 10\% compounded annually, this investment will grow to LE110 in one year. The investment earned LE10. At the end of year two, the current balance LE110 will be invested and this investment will grow to LE121 [110 x (1 + 0.1)].

2.3.1 Simple Interest

Simple interest is calculated using the principal only (i.e. the original investment or original loan), ignoring any interest that has been accrued in preceding interest periods.
Simple interest is seldom used in today’s economics. Thus, if you were to loan a present sum of money \( P \) to someone at a simple annual interest rate \( i \) for a period of \( n \) years, the amount of interest you would receive from the loan would be:

\[
Total \ Interest = Principal \ (P) \times Number \ of \ periods \ (n) \times Interest \ rate \ (i) \\
= P \times n \times i 
\] (2.5)

At the end of \( n \) years the amount of money due \( F \) would equal the amount of the loan \( P \) plus the total interest earned.

\[
F = P + P \times n \times i \\
= P \left(1 + i \times n\right) 
\] (2.6)

Example 2.2: If you borrow LE1,000 for three years at 6% per year simple interest, how much money will you owe at the end of three years?

Solution: Simple Interest = Principal \( (P) \times Number \ of \ periods \ (n) \times Interest \ rate \ (i) \\
= LE1,000 \times 3 \times 0.06 = LE180 \\
Amount \ due \ after \ three \ years = LE1,000 + LE180 = LE1,180

2.3.2 Compound Interest

Compound interest is calculated using the principal plus the total amount of interest accumulated in previous periods. Thus, compound interest means “interest on top of interest”.

Example 2.3: If you borrow LE1,000 at 6% per year compound interest, compute the total amount owed after three year period?

Solution: Interest for Year 1 = LE1,000 \times 0.06 = LE60 \\
Total amount due after year 1 = LE1,000 + LE60 = LE1,060 \\
Interest for year 2 = LE1,060 \times 0.06 = LE63.60 \\
Total amount due after year 2 = LE1,060 + LE63.60 = LE1,123.60 \\
Interest for year 3 = LE1,123.60 \times 0.06 = LE67.42 \\
Total amount due after year 3 = LE1,123.60 + LE67.42 = LE1,191.02 \\
Thus, with compound interest, the original LE1,000 would accumulate an extra LE1,191.02 - LE1,180 = LE11.02 compared to simple interest in the three year period.
Therefore, if an amount of money $P$ is invested at some time $t = 0$, the total amount of money ($F$) that would be accumulated after one year would be:

$$F_1 = P + Pi$$

$$F_1 = P(1 + i)$$

Where $F_1$ = the total amount accumulated after one year

At the end of the second year, the total amount of money accumulated ($F_2$) would be equal to the total amount that had accumulated after year 1 plus interest from the end of year 1 to the end of year 2.

$$F_2 = F_1 + F_1 i$$

$$F_2 = P(1 + i) + P(1 + i)i$$

$$F_2 = P(1 + i)(1 + i)$$

$$F_2 = P(1 + i)^2$$

Similarly, the total amount of money accumulated at the end of year 3 ($F_3$) would be equal to the total amount that had accumulated after year 2 plus interest from the end of year 2 to the end of year 3.

$$F_3 = F_2 + F_2 i$$

$$F_3 = P(1 + i)^2 + P(1 + i)^2 i$$

$$F_3 = P(1 + i)^2(1 + i)$$

$$F_3 = P(1 + i)^3$$

From the preceding values, it is evident that by mathematical induction that the formula for calculating the total amount of money ($F$) after ($n$) number of years, using compound interest ($i$) would be:

$$F = P(1 + i)^n \quad (2.7)$$

This is the “single-payment compound amount formula” and the values between the parenthesis $(1 + i)^n$ is called the “single-payment compound amount factor”. This equation could be written in the functional notation as:
\[ F = P (F/P, i, n) \]  \hspace{1cm} (2.8)

*Equation 2.9 is read as: find a future sum “F” given a present sum “P” at an interest rate “i” per period and “n” interest periods.*

A present sum \( P \) can be determined, easily, knowing a future sum \( F \) by rewriting Equation 2.7 as follows:

\[ P = F / (1 + i)^n = F 
\]

(2.9)

This is the “**single-payment present worth formula**” and the values between the parenthesis \([1 / (1 + i)^n]\) is called the “**single-payment present worth factor**”. This equation could be written in the functional notation as:

\[ P = F (P/F, i, n) \]

(2.10)

*Equation 2.10 is read as: find a present worth sum “P” given a future sum “F” at an interest rate “i” per period and “n” interest periods.*

**Example 2.3:** If a LE500 were deposited in a bank savings account, how much would be in the account three years hence if the bank paid 6% per year compound interest?

*Solution:* \( P = \text{LE500} \quad i = 0.06 \quad n = 3 \quad F = \text{unknown} \)

\[ F = P(1 + i)^n = 500(1 + 0.06)^3 = \text{LE595.5} \]

**Example 2.4:** If you wished to have LE800 in a savings account at the end of 4 years from now, and 5% interest was paid annually. How much would you put into the saving account now?

*Solution:* \( F = \text{LE800} \quad i = 0.05 \quad n = 4 \quad P = \text{unknown} \)

\[ P = F(1 + i)^n = 800(1 + 0.065)^3 = \text{LE658.16} \]

### 2.3.3 Use of Interest Tables

To avoid the trouble of writing out each formula and using calculators, a standard notation is used: \((X/Y, i, n)\). The first letter in the parentheses \((X)\) represent what you “Want to find”, while the second letter \((Y)\) represents what is “Given”. For example, \(F/P\)
means “find F when given P”. The “i” is the interest rate in percent and “n” represents the number of periods involved.

Alternate solution for example 2.3 using the interest tables:
The functional notation is written as \((F/P, i, n) = (F/P, 6\%, 3)\). Knowing \(n = 3\), locate the proper row in the 6% Table under the “Single Payment, Compound Amount Factor” column, 1.191. Thus,

\[
F = P(F/P, 6\%, 3) = 500(1.191) = \text{LE}595.5
\]

Alternate solution for example 2.4 using the interest tables:
The functional notation is written as \((P/F, i, n) = (P/F, 5\%, 4)\). Knowing \(n = 4\), locate the proper row in the 5% Table under the “Single Payment, Present Worth Factor” column, 0.8227. Thus,

\[
P = F(P/F, 5\%, 4) = 800(0.8227) = \text{LE}658.16
\]

2.4 Cash Flow/Time Diagrams
In cash flow diagrams, the amount of \(F\) (future sum of money) and \(A\) (periodic equal payments) are considered at the end of the interest period. Every person or company has cash receipts (income) and cash disbursement (costs). The results of income and costs are called cash flows.

\[
\text{Cash Flow} = \text{Receipts} – \text{Disbursements} \tag{2.11}
\]
A positive cash flow indicates net receipts in a particular interest period or year. A negative cash flow indicates a net disbursement in that period.

Example 2.5: If you buy a printer in 1999 for \text{LE}300, and maintain it for three years at \text{LE}20 per year, and then sell it for \text{LE}50, what are your cash flows for each year?

Solution:
It is important to remember that all receipts and disbursements and thus cash flows are assumed to be end-of period amounts. Therefore, 1999 is the present (now) and 2002 is the end of year 3.

**Example 2.6:** Suppose you borrowed LE1,000 on May 1, 1984, and agree to repay the loan in a sum of LE1,402.60 after four years at 7%. Tabulate the cash flows?

**Solution:**

<table>
<thead>
<tr>
<th>Year</th>
<th>Receipts</th>
<th>Disbursement</th>
<th>Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>May 1, 1984</td>
<td>LE1000</td>
<td>0</td>
<td>1000</td>
</tr>
<tr>
<td>May 1, 1985</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>May 1, 1986</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>May 1, 1987</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>May 1, 1988</td>
<td>0</td>
<td>LE1402.6</td>
<td>-LE1402.6</td>
</tr>
</tbody>
</table>

A cash flow diagram is simply a graphical representation of cash flows (in vertical direction) on a time scale (in horizontal direction). Time zero is considered to be present, and time 1 is the end of time period 1. This cash flow diagram is setup for five years.

The direction of the cash flows (income or outgo) is indicated by the direction of the arrows. From the investor’s point of view, the borrowed funds are cash flows entering the system, while the debt repayments are cash flows leaving the system.
Example 2.7: If you borrow LE2,000 now and must repay the loan plus interest (at rate of 6% per year) after five years. Draw the cash flow diagram. What is the total amount you must pay?

Solution:

F = LE2,000 \times (1+0.06)^5
F = LE2,676.45

Example 2.8: If you start now and make five deposits of LE1,000 per year (A) in a 7% per year account, how much money will be accumulated immediately after you have made the last deposit? Draw the cash flow diagram. What is the total amount you will accumulate?

Solution: Since you have decided to start now, the first deposit is at year zero and the fifth deposit and withdrawal occurs at end of year 4.
Example 2.9: Assume that you want to deposit an amount \( P \) into an account two years from now in order to be able to withdraw LE400 per year for five years starting three years from now. Assume that the interest rate is 5.5% per year. Construct the cash flow diagram.

Solution:

![Cash Flow Diagram](image)

Example 2.10: Suppose that you want to make a deposit into your account now such that you can withdraw an equal amount \( A_1 \) of LE200 per year for the first five years starting one year after your deposit and a different annual amount \( A_2 \) of LE300 per year for the following three years. With an interest rate \( i \) of 4.5% per year, construct the cash flow diagram.

Solution: The first withdrawal (positive cash flow) occurs at the end of year 1, exactly one year after \( P \) is deposited.

![Cash Flow Diagram](image)

2.5 Uniform Series Payments

In many time, there are situations where a uniform series of receipts or disbursements \( A \) will be paid or received. The Future Value, \( F \), of a uniform annual payment, \( A \), is calculated at the end of the period, \( n \), in which the last payment occurs with an investment rate \( i \). Thus, the future value of a five year annual payment is computed at the end of each year. The Future Value of the uniform annual payments is equal to the sum of
the future values of the individual payments at that time.

In the case of \( n \) years:

\[
F = A(1 + i)^{n-1} + A(1 + i)^{n-2} + \ldots + A(1 + i)^3 + A(1 + i)^2 + A(1 + i) + A \quad (2.12)
\]

Multiplying both sides by \((1 + i)\), then

\[
F(1 + i) = A(1 + i)^n + A(1 + i)^{n-1} + \ldots + A(1 + i)^3 + A(1 + i)^2 + A(1 + i) \quad (2.13)
\]

Subtracting Eq. 2.12 from Eq. 2.13, then

\[
F(1 + i) - F = A(1 + i)^n - A
\]

\[iF = A[(1 + i)^n - 1], \text{ then} \]

\[
F = A\left(\frac{(1+i)^n - 1}{i}\right) \quad (2.14)
\]

Using this equation, we could determine \( F \) when \( A \) is known. This equation is named **uniform series compound amount formula**. When using the interest tables, Eq. 2.14 could be represented as:

\[
F = A(F/A, i\%, n)
\]

The term within the brackets is called the **uniform series compound amount factor**. Accordingly, the annual uniform amount, \( A \), to be invested at the end of each period in
order to produce a fixed amount, \( F \), at the end of \( n \) periods with interest rate \( i \) could be calculated as follow from Eq. 2.14.

Equation 2.14 could be rearranged to become as follow:

\[
A = F \left( \frac{i}{(1+i)^n - 1} \right) \tag{2.15}
\]

Using Eq. (2.15), we could determine \( A \) when \( F \) is known. This equation is named \textit{uniform series sinking fund formula}. When using the interest tables, Eq. 2.15 could be represented as:

\[
A = F \left( \frac{A}{F}, i\%, n \right)
\]

The term within the brackets is called the \textit{uniform series sinking fund factor}. Equation 2.15 could be used to convert a future amount of money, \( F \), will be received after \( n \) years into equal annual payments, \( A \).

Now, the present worth, \( P \), of a future amount of money, \( F \), from a uniform series payments, \( A \), could be calculated from Eq. 2.14 as follow:

\[
P = A \left( \frac{(1+i)^n - 1}{i(1+i)^n} \right) \tag{2.16}
\]

Using Eq. (2.16), we could determine \( P \) when \( A \) is known. This equation is named \textit{uniform series present worth formula}. When using the interest tables, Eq. 2.16 could be represented as:

\[
P = A \left( \frac{P}{A}, i\%, n \right)
\]

The term within the brackets is called the \textit{uniform series present worth factor}. Equation 2.16 could be used to determine the annual uniform amount, \( A \), to be invested at the end of \( n \) periods with interest rate \( i \) to produce a present worth, \( P \).

Using the uniform series present worth formula (Eq. 2.16), the value of a uniform series
payment, \( A \), when the present sum, \( P \), is known could be determined by rearranging this equation as follow:

\[
A = P \left( \frac{i(1+i)^n}{(1+i)^n - 1} \right)
\]  \hspace{1cm} (2.17)

Using Eq. (2.17), we could determine \( A \) when \( P \) is known. This equation is named \textit{uniform series capital recovery formula}. When using the interest tables, Eq. 2.17 could be represented as:

\[
A = P(A/P, i\%, n)
\]

The term within the brackets is called the \textit{uniform series capital recovery factor}. Equation 2.17 could be used to determine the annual uniform amount, \( A \), to be invested at the end of \( n \) periods with interest rate \( i \) to produce a present worth, \( P \).

\textit{Example 2.11}: On January 1, a man deposits LE5000 in a bank that pays 8\% interest, compounded annually. He wishes to withdraw all the money in five equal end-of-year sums beginning December 31\textsuperscript{st} of the first year. How much should he withdraw each year?

\textit{Solution}: \( P = \text{LE}5000; \quad n = 5; \quad i = 8\%; \quad A = \text{unknown} \)

\[
A = 5000 \left( \frac{0.08 \times 1.08^5}{1.08^5 - 1} \right) = \text{LE}1252
\]

\textit{Example 2.12}: A man deposits LE500 in a bank at the end of each year for five years. The bank pays 5\% interest, compounded annually. At the end of five years, immediately following his fifth deposit, how much will he have in his account?

\textit{Solution}: \( A = \text{LE}500; \quad n = 5; \quad i = 5\%; \quad F = \text{unknown} \)

\[
F = 500 \left( \frac{1.08^5 - 1}{0.05} \right) = \text{LE}2763
\]

\textit{Example 2.13}: If a person deposits LE600 now, and LE300 two year from now, and LE400 five years from now, how much will he have in his account ten years from now if the interest rate is 5\%?
Solution:

\[ F = LE600 \left( \frac{F}{P}, 5\%, 10 \right) + LE300 \left( \frac{F}{P}, 5\%, 8 \right) + LE400 \left( \frac{F}{P}, 5\%, 5 \right) \]
\[ = LE600(1.6289) + LE300(1.4774) + LE400(1.2763) = LE1931.08 \]

**Example 2.14:** How much money would a person have after eight years if he deposited LE100 per year for eight years at 4% starting one year from now?

**Solution:**

\[ F = LE100 \left( \frac{F}{A}, 4\%, 8 \right) = LE100(9.214) \]
\[ = LE921.40 \]

**Example 2.15:** How much money would you be willing to spend now in order to avoid spending LE500 seven years from now if the interest rate is 4.5%?

**Solution:**

\[ P = LE500 \left( \frac{P}{F}, 4.5\%, 7 \right) = LE500(0.7353) = LE367.65 \]
Example 2.16: How much money would you be willing to pay now for a note that will yield LE600 per year for nine years if the interest rate is 7%?

Solution:

\[ P = \text{LE}600 \left( \frac{P}{A}, 7\%, 9 \right) = \text{LE}600(6.5152) = \text{LE}3909.12 \]

Example 2.17: How much money must a person deposit every year starting one year from now at 5.5% per year in order to accumulate LE6,000 after seven years?

Solution:

\[ A = \text{LE}6,000 \left( \frac{A}{F}, 5.5\%, 7 \right) = \text{LE}6,000(0.12098) \]
\[ = \text{LE}725.88 \text{ per year} \]

Example 2.18: A couple wishing to save money for their child’s education purchased an insurance policy that will yield LE10,000 15 years from now. The parent must pay LE500 per year for the 15 years starting one year from now. What will be the rate of return on their investment?

Solution:
\[ A = F(A/F, i\%, n) \]

\[ \text{LE}500 = \text{LE}10,000(A/F, i\%, 15) \]

From the interest tables under the \( A/F \) column for 15 years, the value of 0.0500 is found to lie between 3\% and 4\%.

\((A/F, i\%, 15) = 0.0500\)

By interpolation, \( i = 3.98\% \)

**Example 2.19:** How long would it take for \( \text{LE}1,000 \) to double if the interest rate is 5\%?

**Solution:**

\[ P = F(P/F, i\%, n) \]

\[ \text{LE}1000 = \text{LE}2,000(P/F, 5\%, n) \]

\((P/F, 5\%, n) = 0.500\)

From the 5\% interest table, the value of 0.500 under the \( P/F \) column lies between 14 and 15 years. By interpolation, \( n = 14.2 \) years

### 2.6 Multiple Factors

When a uniform series of payment \( (A) \) begins at a time other than the end of year 1, several methods can be used to find the present worth \( (P) \). For example, given:

We can use the following methods:

- Use the single-payment present worth factor \( (P/F, i\%, n) \) to find the present worth of each disbursement at year Zero, and then add them.
- Use the single-payment compound-amount factor \((F/P, i\%, n)\) to find the future worth of each disbursement in year 13, add them, and then find the present worth of the total using \(P = F(P/F, i\%, 13)\).

- Use the uniform-series compound-amount factor \((F/A, i\%, n)\) to find the future amount by \(F = A(F/A, i\%, 10)\) and then find the present worth using \(P = F(P/F, i\%, 13)\).

- Use the uniform-series present-worth factor \((P/A, i\%, n)\) to compute the present worth at year 3 and then find the present worth in year Zero by using the \((P/F, i\%, n)\) factor.

**Note:** It is very important to remember that the present worth is always located One year prior to the first annual payment when using the uniform-series present-worth factor \((P/A, i\%, n)\). On the other hand, the uniform-series compound-amount factor \((F/A, i\%, n)\) was derived with the future worth “F” located in the same year as the last payment. It is always important to remember that the number of years \(n\) that should be used with the \(P/A\) or \(F/A\) factors is equal to the number of payments. It is generally helpful to re-number the cash-flow diagram to avoid counting errors.

**Example 2.20:** A person buys a piece of property for LE5,000 down-payment and deferred annual payments of LE500 a year for six years starting three years from now. What is present worth of the investment if the interest rate is 8%?

**Solution:**

\[
P_{Ai} = \text{the present worth of a uniform-series.}
\]

\[
P_A = \text{the present worth at a time other than zero}
\]
Example 2.21: Calculate the eight-year equivalent uniform annual series at 6% interest for the uniform disbursements shown in the figure below.

Solution:

\[ P_{A1} = 800 \left( \frac{P}{A}, 6\%, 6 \right) = \text{LE}3933.6 \]
\[ P_{T} = 3933.6 \left( \frac{P}{F}, 6\%, 2 \right) = \text{LE}3,501.12 \]
\[ A' = P_{T} \left( \frac{A}{P}, 6\%, 8 \right) = \text{LE}563.82 \]

Example 2.22: Calculate the present worth \((P)\)?

Solution:

Find the present worth of the uniform-series and add it to the present-worth of the two individual payments:

\[ P = \text{LE20,000} \left( \frac{P}{A}, 6\%, 20 \right) + \text{LE10,000} \left( \frac{P}{F}, 6\%, 6 \right) + \text{LE15,000} \left( \frac{P}{F}, 6\%, 16 \right) = \text{LE}242,352 \]
Example 2.23: Calculate the equivalent uniform annual series \( (A') \)?

Solution: Two methods could be used:

Present worth method
\[
A' = LE20,000 + LE10,000(P/F, 6\%, 6) (A/P, 6\%, 20) + LE15,000(P/F, 6\%, 16) (A/P, 6\%, 20) = LE21,129 \text{ per year}
\]

Future worth method
\[
A = LE20,000 + LE10,000(F/P, 6\%, 14) (A/F, 6\%, 20) + LE15,000(F/P, 6\%, 4) (A/F, 6\%, 20) = LE21,129 \text{ per year}
\]

Example 2.24: Calculate the present worth of the following series of cash flows if \( i = 8\% \)?

<table>
<thead>
<tr>
<th>year</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>+</td>
<td>LE460</td>
<td>+</td>
<td>LE460</td>
<td>+</td>
<td>LE460</td>
<td>+</td>
<td>LE460</td>
</tr>
</tbody>
</table>

Solution:

\[
P = P_t + P_A - P_F
\]
\[
P = LE460 + LE460(P/A, 8\%, 6) - LE5,000(P/F, 8\%, 7)
\]
\[
= -LE331
\]
2.7 Uniform Infinite Series

Civil engineering projects (roads, bridges, dams, etc.) are generally designed and constructed to last for long periods that may exceed 100 years. These projects are considered as long-aged projects. In the uniform infinite series payments, uniform payments are invested at the end of each period for a very long time that may be considered as infinite times. This could be represented as follows:

\[ A' \]

The infinite uniform series payments are represented as \( A' \) and the present value is presented as \( P' \). To calculate \( P' \), Eq. (2.16) could be re-written as follows:

\[
P = \frac{A}{i} \left[ \frac{(1+i)^n - 1}{(1+i)^n} \right] = \frac{A}{i} \left[ 1 - \frac{1}{(1+i)^n} \right]
\]

Substituting for \( n \) as infinite number (\( \infty \)), then

\[ P' = \frac{A'}{i} \]  \hspace{1cm} (2.18)

Accordingly, we could calculate the uniform infinite series payments \( A' \) considering Eq. (2.18) as follows:

\[ A' = P' \times i \]  \hspace{1cm} (2.19)

**Example 2.25:** The construction of a bridge costs LE10,000,000, the annual maintenance cost LE40,000 at the first 10 years. Then, the maintenance cost increases to LE50,000 after that. It is required to calculate the present worth of the costs and the equivalent annual cost if the interest rate is 8%?

**Solution:**

The cash flow curve is represented as follows:
\[ P'T = P + P1 + P2 \]
\[ P1 = 40,000(P/A, 8\%, 10) = LE268404 \]
\[ P' = A' / i = 50,000 / 0.08 = LE625,000 \]
\[ P2 = 625,000(P/F, 8\%, 10) = 625,000 (0.4632) = LE289,500 \]
\[ PT = 10,000,000 + 268,404 + 289,500 = LE10,557,904 \]

### 2.8 Arithmetic Gradient Uniform Series Payments

In case of the cash flow or payments is not of constant amount \( A \), there is a uniformly increasing series. The uniformly increased payments may be resolved into two components as shown below.
In this case, \( P = P' + P'' \)

The arithmetic gradient is a series of increasing payments as shown above. It could be dealt with as a series of individual payments. The value of “\( F \)” for the sum of all payments at time \( n \) is:

\[
F = G(1 + i)^{n-2} + 2G(1 + i)^{n-3} + \ldots + (n - 2)(1 + i) + (n - 1)G 
\]  
(2.20)

\[
F = G[(1 + i)^{n-2} + 2(1 + i)^{n-3} + \ldots + (n - 2)(1 + i) + n - 1] 
\]  
(2.21)

Multiplying both sides by \((1 + i)\)

\[
(1 + i)F = G[(1 + i)^{n-1} + 2(1 + i)^{n-2} + \ldots + (n - 2)(1 + i)^2 + (n - 1)(1 + i)] 
\]  
(2.22)

Subtracting Eq. (2.21) from Eq. (2.22), yields:

\[
iF = G[(1 + i)^{n-1} + (1 + i)^{n-2} + \ldots + (1 + i)^2 + (1 + i) + 1] - nG 
\]  
(2.23)

From the derivation of Eq. 2.14, the term between the brackets was proven to equal: 
\([(1+i)^n-1/i]\). Thus, Eq. 2.23 could be written as:

\[
iF = G\left[\frac{(1+i)^n-1}{i}\right] - nG 
\]  
(2.24)

Then,

\[
F = \frac{G}{i}\left[\frac{(1+i)^n-1}{i} - n\right] 
\]  
(2.25)

Accordingly, the present worth of \( F \) could be determined by dividing Eq. 2.25 by \((1+i)^n\).

\[
P = G\left[\frac{(1+i)^n-in-1}{i^2(1+i)^n}\right] 
\]  
(2.26)

Using Eq. (2.26), we could determine \( P \) when \( G \) is known. When using the interest tables, Eq. 2.26 could be represented as:

\[
P = G(P/G, i\%, n) 
\]

The term within the brackets is called the arithmetic gradient present worth factor.
Equation 2.26 could be used to determine the present worth amount, \( P \), of a gradient payments, \( G \), for \( n \) periods with interest rate \( i \).

Multiplying Eq. 2.25 by the sinking fund factor, \([1/(1+i)^n-1]\), we could determine the uniform series payments, \( A \), from arithmetic gradient payments, \( G \).

\[
A = \left( (1+i)^n - in - 1 \right) \left( \frac{1}{i} \left(1 + \frac{n}{1+n} \right) \right)
\]

(2.27)

Using Eq. (2.27), we could determine \( A \) when \( G \) is known. When using the interest tables, Eq. 2.27 could be represented as:

\[
A = G \left( \frac{A}{G}, i\%, n \right)
\]

The term within the brackets is called the **arithmetic gradient uniform series factor**.

**Example 2.26**: A man purchased a new automobile. He wishes to set aside enough money in a bank account to pay the car maintenance for the first five years. It has been estimated that the maintenance cost of an automobile is as follows:

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost (LE)</td>
<td>120</td>
<td>150</td>
<td>180</td>
<td>210</td>
<td>240</td>
</tr>
</tbody>
</table>

Assume the maintenance costs occur at the end of each year and that the bank pays 5% interest. How much he should deposit in the bank now.

**Solution**: The cash flow may be broken into two components as shown below.
The first is uniform series present worth and the second is arithmetic gradient series present worth. Note that the value of n in the gradient series payments is 5 not 4 where there are 4 terms containing $G, (n - 1)$.

$$P = 120(P/A, 5\%, 5) + 30(P/G, 5\%, 5)$$

$$= 120(4.329) + 30(8.237) = 59 + 247 = LE766$$

Example 2.27: Calculate the equivalent uniform annual cost ($A$) of the following schedule of payments.

![Cash Flow Diagram](image)

Solution: Since payments repeat every five years, analyze for 5 years only. In this example, $G = LE100$

$$A = 100 + 100(A/G, 8\%, 5) = LE284.60$$

Example 2.28: The uniform equivalent of the cash flow diagram shown is given by which one of the following five answers?

(a) $50(A/G, i, 8)$

(b) $50(A/G, i, 9)$

(c) $50(A/G, i, 10)$

(d) $50(A/G, i, 9)(F/A, i, 9)(A/F, i, 10)$

(e) $50(P/G, i, 8)(P/F, i, 1)(A/P, i, 10)$

Solution: Note these two concepts:

1) The $G$ series is 9 periods long

2) The uniform equivalent is 10 periods long
The answer is (d)

**Example 2.29:** Find the Present Equivalent of the following cash flow if \( i = 18\% \).

![Cash Flow Diagram]

**Solution:** The original cash flow is represented as follow:

\[
P1 = 100 + 150(P/A, 18\%, 10) = LE774.10 \\
P2 = 50(P/G, 18\%, 10) = LE717.60 \\
P3 = 100(P/G, 18\%, 6)(P/F, 18\%, 4) = LE365.34 \\
P = P1 + P2 - P3 = LE1,126.36
\]

**Example 2.30:** A couple wants to begin saving money for their child's education. They estimate that LE10,000 will be needed on the child's 17\(^{th}\) birthday, LE12,000 on the 18\(^{th}\) birthday, LE14,000 on the 19\(^{th}\) birthday, and
LE16,000 on the 20\textsuperscript{th} birthday. Assume an 8\% interest rate with only annual compounding. The couple is considering two methods of setting aside the needed money.

(a) How much money would have to be deposited into the account on the child's first birthday (note: a child's "first birthday" is celebrated one year after the child is born) to accumulate enough money to cover the estimated college expenses?

(b) What uniform annual amount would the couple have to deposit each year on the child's first birth through seventeenth birthdays to accumulate enough money to cover the estimated college expenses?

\textit{Solution:}

\textbf{a)}

\[
\text{note: year zero corresponds to child's 1\textsuperscript{st} birthday}
\]

\[
F = P(1 + i)^n = P(1 + 0.08)^4
\]

\[
F = P \times 1.3605
\]

Let \( F = \) the LE's needed at the beginning of year 16

\[
F = 10,000(P/A, 8\%, 4) + 2,000(P/G, 8\%, 4)
\]

\[
F = 42,420
\]

The amount needed today \( P = 42,420(P/F, 8\%, 15) = 13,370.80 \)

\textbf{b)}

\[
P' = 13,370.80(P/F, 8\%, 1) = 12,380.00
\]

\[
A = 12,380.00(A/P, 8\%, 17) = 1,356.85
\]
2.9 Uniform Series Infinite Payments Every \( t \) Period

In this case, equal payments, \( A' \), are invested at the end of equal periods, \( t \) (not one year) for an infinite time. In this case, the interest rate is given annually while the periods are given in different times (e.g., 4 years). The following equation is used:

\[
P = \frac{A'}{(1+i)^t - 1}
\]

**Example 2.31:** The maintenance of a bridge costs LE15,000 every 3-years. Calculate the equivalent uniform annual cost if the interest rate is 5%. Consider this is an aged project.

**Solution:** Using Eq. 2.28, the present worth could be determined.

\[
P = \frac{15000}{(1 + 0.05)^3 - 1} = \text{LE95,162}
\]

Then, \( A' = P'r = 95,162 \times (0.05) = \text{LE4,758} \)

2.10 Nominal and Effective Interest

**Example 2.32:** Consider the situation of a person depositing LE100 into a bank that pays 5% interest, compounded semi-annually. How much would be this amount at the end of one year?

**Solution:** Five percent interest, compounded semi annually, means that the pays 2.5% every six month. This, the initial amount \( P = \text{LE100} \) would be at the end of 6 month: \( 100(1 + 0.025) = \text{LE102.5} \) and the end of the second 6 month, this amount will be \( = 102.5(1.025) = \text{LE105.06} \) or, this could be calculated as follows:

\[
F = 100(1 + 0.025)^2 = \text{LE105.06}
\]

*Nominal interest rate per year, \( r \), is the annual interest rate without considering the effect of any compounding. While, effective interest rate per year, \( i_a \), is the annual interest rate taking into account the effect of any compounding during the year.*

To find the value of the effective interest rate, assume the following:
\[ r = \text{Nominal interest year per period (usually one year)}. \]
\[ i = \text{Effective interest rate per period}. \]
\[ i_a = \text{Effective interest rate per year}. \]
\[ m = \text{Number of compounding sub-periods per time period}. \]

Then, \[ i = \frac{r}{m} \]

The future value, \( F \), of \( P \) considering \( i_a \) is:
\[ F = P(1 + i_a) \]

While using \( i \) compounded over all periods, \( m \):
\[ F = P(1 + i)^m \]

Consider of the \( F \) value for a present worth \( P \) of LE1 and equating the two expressions for \( F \) and substituting LE1 for \( P \):
\[ 1 + i_a = (1 + i)^m \]

Then: \[ i_a = (1 + i)^m - 1 \text{ or } i_a = (1 + \frac{r}{m})^m - 1 \]

Solving for the effective interest rate, then:
\[ i = (1 + i_a)^{1/m} - 1 \quad (2.29) \]

**Example 2.33:** Consider \( r = 13\% \) compounded monthly. Find effective interest rate per year.

**Solution:**
\[ r = 13; \text{ then, } i = \frac{r}{m} = \frac{13}{12} = 1.08333\% = 0.018333 \]
\[ i_a = (1 + i)^m - 1 = (1 + 0.018333)^{12} - 1 = 13.80\% \text{ per year} \]

**Example 2.34:** Given nominal and effective rates of 16 and 16.986 \%. What is the compounding period?

**Solution:**
\[ 16.986 = (1 + 16/m)^m - 1 \]
Solving for \( m \) by trial and error, then \( m = 4 \), therefore quarterly.
Example 2.35: Given an interest rate of 1% per month. What is the equivalent effective rate per 2 months?

Solution:
Effective $i$ for 2 moths = $(1 + 0.01)^2 - 1$
Effective $i = 0.0201 = 2.01\%$ for 2 months.

Example 2.36: A bank lends money on the following terms: “If I give you LE50 on Monday, you owe me LE60 on the following Monday.” What nominal interest rate per year ($r$) this bank charging? What effective interest rate per year ($i_a$) is the charging? If the bank started with LE50 and was able to keep it, as well as the money received, out in loans at all times, how much money would have at the end of one year?

Solution:

$60 = 50(1 + r)$

$(1 + r) = 1.2$; then $r = 0.2 = 20\%$ per week

Then, nominal interest rate per year = $52 \text{ weeks} \times 0.2 = 10.4 = 1040\%$

Effective interest rate per year, $i_a = (1 + r/m)^m - 1 = (1 + 0.2)^{52} - 1$

$= 13104 = 1310400\%$

$F = P(1 + i)^n = 50(1 + 0.2)^{52} = \text{LE655,200}$

2.11 Exercises

1. Find the future value of LE10 deposit earning simple interest of 7.6\% per year at the end of the 5$^{th}$ year.

2. In how many years, LE100 will become LE260 if $I = 0.5\%$ (simple interest) per year.

3. If we deposit LE1,000 in a bank at 7\% compounded annually for 10 years. How much you will have? How much must we put in the bank now at 9%/year to accumulate LE5,000 in 8 years?
4. A tax refund expected one year from now has a present worth of LE3000 if \( i = 6\% \). What is its present worth if \( i = 10\% \)?

5. What sum of money now is equivalent to LE8250 two years from now, if interest rate is 8\% per annum, compounded semi-annually?

6. If the population of Mansoura City is currently 800,000 and the annual constant growth rate is estimated to be 10\%, what will the population be in 10 years from now?

7. Ahmed contracted with a car dealer on a car on 4-year basis for LE100,000. An initial deposit of LE34,000 paid at the time of contract and the rest will be paid at the time of delivery (4 years from now). How much he should deposit now? Assume \( i = 10\% \) compounded annually.

8. How long would it take any sum to triple itself at a 5\% annual interest rate?

9. Given a sum of money \( Q \) that will be received six years from now. At 5\% interest rate, the present worth now of \( Q \) is LE60. At this same interest rate, what would be the value of \( Q \) ten years from now?

10. Consider the following situation:

<table>
<thead>
<tr>
<th>Year</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash flow</td>
<td>(+P)</td>
<td>0</td>
<td>0</td>
<td>-400</td>
<td>0</td>
<td>-600</td>
</tr>
</tbody>
</table>

Solve for \( P \) assuming 12\% interest rate compounded annually.

11. On January 1\(^{st}\), a man deposited LE5,000 in a credit union that pays 8\% interest annually. He wishes to withdraw all the money in five equal end-of-year sums, beginning December 31\(^{st}\) of the first year. How much should he withdraw each year?
12. You have been given a monopoly to sell a product in an area for a period of 5 years. You expect to receive a profit of LE100,000 per year over this period. After 5 years other competitors will enter the region and your expected profit will drop in half. Five years later the product will be obsolete and your business will close. If interest rate is 7% for all 10 years, what is the present value of your total profit?

13. A proposed investment in an assembly line will have an initial purchase and installation cost of LE175,000. The annual maintenance cost will be LE6000; periodic overhauls once every 3 years, excluding the last years of use will cost LE11,500 each. The improvement will have a useful life of 9 years. What is the present worth of the 9-year costs of the improvement at $i=8\%$?

14. A friend offers you his train tickets (you really want these tickets) for LE250. You say that you don’t have that much right now, but you could pay him LE50.84 every 2 weeks for the next 12 weeks. He says okay.
   (a) What is the nominal interest rate that you will be paying?
   (b) What is the effective interest rate that you will be paying?

15. Suppose that you recently inherited LE20,000, and you decide to invest it and also start making additional payments each month toward your retirement. How much would you have to deposit every month in order to attain your goal of LE3,000,000 at the end of 30 years? Assume that you can reliably earn 1% per month during this period on your investments. (Assume that your total investment consists of the amount you inherited as well the additional monthly amounts that you are putting in the fund).

16. A steel bridge costs $450,000 to build and $12,000 per year for maintenance. The bridge deck will be resurfaced every 10 years for $290,000, and anticorrosion paint will be applied every 2 years for $28,000. Assume that the bridge has a design life of 60 years. The interest rate is 8% per annum. Determine the EAC for
the bridge, assuming that the bridge will neither be resurfaced nor repainted at the end of the 60th year.

17. A contractor offers to purchase your old tractor for LE10,000 but cannot pay you the money for 12 months. If you feel $i=1\%$ per month is a fair interest rate, what is the present worth to you today of the $10,000 if it is paid 12 months from now.

18. Determine the effective interest rate for a nominal annual rate of 6% that is compounded:
   a. Semiannually
   b. Quarterly
   c. Monthly
   d. Daily (assuming 365 days a year)

19. Solve for the value of $X$ in the figure below so that cash flow A is equivalent to cash flow B. Assume a per period interest rate of 12%.

![Cash flow diagram]

20. You decided to buy your house with a mortgage of LE300,000. The mortgage’s yearly interest rate is 8%, and will be paid off in 30 equal annual payments. After the 12th payment, you have the opportunity to refinance your balance at a yearly rate of 6%, to be paid off in 40 equal annual payments. What will your new annual payments be?
21. The investment in a crane is expected to produce profit from its rental as shown below, over the next six years. What is the present worth of the investment, assuming 12% interest?

<table>
<thead>
<tr>
<th>Year</th>
<th>Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>LE15,000</td>
</tr>
<tr>
<td>2</td>
<td>LE12,500</td>
</tr>
<tr>
<td>3</td>
<td>LE10,000</td>
</tr>
<tr>
<td>4</td>
<td>LE7,500</td>
</tr>
<tr>
<td>5</td>
<td>LE5,000</td>
</tr>
<tr>
<td>6</td>
<td>LE2,500</td>
</tr>
</tbody>
</table>

22. The annual income from an apartment house is LE20,000. The annual expense is estimated to be LE2,000. If the apartment could be sold for LE100,000 at the end of 10 years, how much could you afford to pay for it now, with 10% considered a suitable interest rate?

23. A municipality is seeking a new tourist attraction, and the town council has voted to allocate LE500,000 for the project. A survey shows that an interesting cave can be enlarged and developed for a contract price of LE400,000. It would have an infinite life. The estimated annual expenses of operation are:
   - Direct Labor LE30,000
   - Maintenance LE15,000
   - Electricity LE5,000

   The price per ticket is to be based upon an average of 1000 visitors per month. If money is worth 8%, what should be the price of each ticket?

24. Consider the following cash flow curve, find the equivalent annuity?
25. A resident will give money to his town to purchase a memorial statue and to maintain it at a cost of LE500 per year forever. If an interest rate of 10% is used, and the resident gives a total of LE15,000; how much can be paid for the statue?

26. What is the Present Worth of a series that decreases uniformly, by LE20 per year, from LE400 in Year 11 to LE220 in Year 20, if \( i \) equal 10%?

27. A project has a first cost of $10,000, net annual benefits of $2000, and a salvage value of $3000 at the end of its 10 year useful life. The project will be replaced identically at the end of 10 years, and again at the end of 20 years. What is the present worth of the entire 30 years of service if the interest rate is 10%.