Analytical Hierarchy process

Introduction

- In our complex world system, we are forced to cope with more problems than we have the resources to handle.

- We need a framework that enable us to think of complex problems in a simple way.

- The AHP provides such a framework that enables us to make effective decisions on complex issues by simplifying and expediting our natural decision-making processes.
## Analytical Hierarchy process

### What is AHP?

- The Analytic Hierarchy Process (AHP) is "A systematic approach to assign weights and priorities to the decision making process."

### Problem:
- Objectives are more important than others
- Objectives are related
- Objectives conflict with each other

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## Analytical Hierarchy process

### What is AHP?


- The AHP is designed to solve complex problems involving multiple criteria

- An advantage of the AHP is that it is designed to handle situations in which the subjective judgments of individuals constitute an important part of the decision process.
Analytical Hierarchy process

What is AHP?

- Basically the AHP is a method of:
  - Breaking down a complex, unstructured situation into its component parts and arranging these parts, or variables into a hierarchic order;
  - Assigning numerical values to subjective judgments on the relative importance of each variable; and
  - Synthesizing the judgments to determine which variables have the highest priority and should be acted upon to influence the outcome of the situation.

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What is AHP?

- The process requires the decision maker to provide judgments about the relative importance of each criterion and then specify a preference for each decision alternative on each criterion.
- The output of the AHP is a prioritized ranking indicating the overall preference for each of the decision alternatives.
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AHP Major Steps

1. Develop a graphical representation of the problem in terms of the overall goal, the criteria, and the decision alternatives. (i.e., the hierarchy of the problem)
2. Specify judgments about the relative importance of each criterion in terms of its contribution to the achievement of the overall goal
3. Indicate a preference or priority for each decision alternative in terms of how it contributes to each criterion
4. Given the information on relative importance and preferences, a mathematical process is used to analyze the information and provide a priority ranking of all alternatives in terms of their overall preference.

AHP Implementation Steps

1. Develop a hierarchy of decision attributes
2. Conduct a pair-wise comparison among attributes
3. Calculate weights/priorities
4. Check for consistency
5. Repeat steps (2) to (4) for each level of attributes in the hierarchy
6. Repeat steps (2) to (4) for each comparison of all alternatives with respect to each attribute
7. Calculate the composite priorities and rank weights
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Hierarchy Development

- The first step in the AHP is to develop a graphical representation of the problem in terms of the overall goal, the criteria, and the decision alternatives.

Overall Goal: Select the Best Car

Criteria: Price, MPG, Comfort, Style

Decision Alternatives: Car A, Car B, Car C

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Hierarchy Development: Example

Attributes: Bid Amount, Managerial Capabilities, Financial Capabilities, Project Duration, Subs Relation

Alternatives: Contractor A, Contractor B, Contractor C, Contractor D
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Pairwise Comparisons

- Pairwise comparisons are fundamental building blocks of the AHP.
- The AHP employs an underlying scale with values from 1 to 9 to rate the relative preferences for two items.

\[
\begin{bmatrix}
C_{11} & C_{1j} & \ldots & C_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & & \vdots \\
C_{n1} & C_{nj} & \ldots & C_{nn}
\end{bmatrix}
\]

Analytical Hierarchy process

Pairwise Comparisons

<table>
<thead>
<tr>
<th>Verbal Judgment of Preference</th>
<th>Numerical Rating</th>
</tr>
</thead>
<tbody>
<tr>
<td>Extremely preferred</td>
<td>9</td>
</tr>
<tr>
<td>Very strongly to extremely preferred</td>
<td>8</td>
</tr>
<tr>
<td>Very strongly preferred</td>
<td>7</td>
</tr>
<tr>
<td>Strongly to very strongly preferred</td>
<td>6</td>
</tr>
<tr>
<td>Strongly preferred</td>
<td>5</td>
</tr>
<tr>
<td>Moderately to strongly preferred</td>
<td>4</td>
</tr>
<tr>
<td>Moderately preferred</td>
<td>3</td>
</tr>
<tr>
<td>Equally to moderately preferred</td>
<td>2</td>
</tr>
<tr>
<td>Equally preferred</td>
<td>1</td>
</tr>
</tbody>
</table>
Analytical Hierarchy process

Pairwise Comparisons

- Element $C_{i,j}$ of the matrix is the measure of preference of the item in row $i$ when compared to the item in column $j$

- AHP assigns a 1 to all elements on the diagonal of the pairwise comparison matrix (comparing an alternative against itself, the judgment must be that they are equally preferred)

- AHP obtains the preference rating of $C_{j,i}$ by computing the reciprocal (inverse) of $C_{i,j}$ (the transpose position)

- The preference value of 2 is interpreted as indicating that alternative $i$ is twice as preferable as alternative $j$

- Thus, it follows that alternative $j$ must be one-half as preferable as alternative $i$

- According above rules, the number of entries actually filled in by decision makers is $(n^2 - n)/2$, where $n$ is the number of elements to be compared.
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Pairwise Comparisons

➢ To begin the pairwise comparison, start at the top of the hierarchy to select the criterion (or, goal, property, attribute), that will be used for making the first comparison. Then, from the level immediately below, take the elements to be compared: A₁, A₂, A₃, and so on.

➢ To compare elements, ask: How much more strongly does this element (or activity) possess (or contribute to, dominate, influence, satisfy, or benefit) the property than does the element with which it is being compared?

Example

Selecting the most appropriate excavation method

Safety

Soil condition

Cost

Underground water condition

Excavation depth

Condition of adjacent building

Construction cost

Construction progress rate

Crossing with existing utilities

Diaphragm wall method

Secant pile method

Soldier pile method

Sheet pile method

Contiguous piling method
Analytical Hierarchy process

Pairwise Comparisons

Example

With respect to selection the most appropriate excavation method

Q1. How important is safety when it is compared to cost?

With respect to selection main criteria “safety”
Q1. How important is soil condition when it is compared to underground water condition?
Q2. How important is soil condition when it is compared to excavation depth?
Q3. How important is soil condition when it is compared to condition of adjacent buildings?
Q4. How important is underground water condition when it is compared to condition of excavation depth?
Q5. How important is underground water condition when it is compared to condition of adjacent buildings?
Q6. How important is excavation depth when it is compared to condition of adjacent buildings?

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Pairwise Comparisons: Example

Municipality Objectives

Costs  LOS  Safety  Environment  Social Impact

Policy A

- Inexpensive
- Good LOS and Safety
- Bad env. Impact
- Very bad Social Impact

Policy B

- Moderately expensive
- Good Safety
- No env. Impact
- Bad LOS and social impact

Policy C

- Very expensive
- Very Good LOS, social and Safety
- Very bad env. Impact
Analytical Hierarchy process

Pairwise Comparisons: Example

- The environment is 3 times as important as LOS, 7 times more important than costs 6 times more important than social impact and equally important as safety.

<table>
<thead>
<tr>
<th></th>
<th>Environment</th>
<th>LOS</th>
<th>Safety</th>
<th>Costs</th>
<th>Social Impact</th>
</tr>
</thead>
<tbody>
<tr>
<td>Environment</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>LOS</td>
<td>0.333333</td>
<td>1</td>
<td>0.5</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>Safety</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Costs</td>
<td>0.142857</td>
<td>0.2</td>
<td>0.2</td>
<td>1</td>
<td>0.25</td>
</tr>
<tr>
<td>Social Impact</td>
<td>0.166667</td>
<td>0.25</td>
<td>0.2</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

Calculate the Priority Weights

- Involves calculating the Eigen vector of the matrix. Two approximate methods that can be used:
  - Column normalization
  - Nth root
Analytical Hierarchy process

Calculate the Priority Weights

- Column Normalization

- Step 1: Sum the values in each column of the comparison matrix

- Step 2: Divide each element in the matrix by its column total
  - The resulting matrix is referred to as the normalized pairwise comparison matrix

- Step 3: Compute the average of the elements in each row of the normalized matrix
  - These averages provide an estimate of the relative priorities of the elements being compared

<table>
<thead>
<tr>
<th></th>
<th>Env</th>
<th>LOS</th>
<th>Safety</th>
<th>Costs</th>
<th>Social Impact</th>
<th>Column Vector</th>
<th>Weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>Environment</td>
<td>0.38</td>
<td>0.47</td>
<td>0.34</td>
<td>0.32</td>
<td>0.37</td>
<td>1.88</td>
<td>0.38</td>
</tr>
<tr>
<td>LOS</td>
<td>0.13</td>
<td>0.16</td>
<td>0.17</td>
<td>0.23</td>
<td>0.25</td>
<td>0.93</td>
<td>0.19</td>
</tr>
<tr>
<td>Safety</td>
<td>0.38</td>
<td>0.31</td>
<td>0.34</td>
<td>0.23</td>
<td>0.31</td>
<td>1.57</td>
<td>0.31</td>
</tr>
<tr>
<td>Costs</td>
<td>0.05</td>
<td>0.03</td>
<td>0.07</td>
<td>0.05</td>
<td>0.02</td>
<td>0.21</td>
<td>0.04</td>
</tr>
<tr>
<td>Social Impact</td>
<td>0.06</td>
<td>0.04</td>
<td>0.07</td>
<td>0.18</td>
<td>0.06</td>
<td>0.41</td>
<td>0.08</td>
</tr>
<tr>
<td>Sum</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>5.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>
Analytical Hierarchy process

Calculate the Priority Weights

- Multiply the $n$ elements in each row by one another and take the $n$th root to form a new column vector.

- Find the priority vector $[W]$ by normalizing the column vector.

<table>
<thead>
<tr>
<th></th>
<th>Env.</th>
<th>LOS</th>
<th>Safety</th>
<th>Costs</th>
<th>Social Impact</th>
</tr>
</thead>
<tbody>
<tr>
<td>Env.</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>LOS</td>
<td>0.333</td>
<td>1</td>
<td>0.5</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>Safety</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Costs</td>
<td>0.1428</td>
<td>0.2</td>
<td>0.2</td>
<td>1</td>
<td>0.25</td>
</tr>
<tr>
<td>Social Impact</td>
<td>0.1666</td>
<td>0.25</td>
<td>0.2</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Product</td>
<td>126.00</td>
<td>2.63</td>
<td>0.38</td>
<td></td>
<td>0.38</td>
</tr>
<tr>
<td>Nth root</td>
<td>2.63</td>
<td>1.27</td>
<td>0.32</td>
<td></td>
<td>0.07</td>
</tr>
</tbody>
</table>

Check Consistency

- An important consideration in terms of the quality of the decision relates to the consistency of judgments of the decision maker during the pairwise comparisons.

- Perfect consistency is very difficult to achieve and some lack of consistency is expected in almost any set of pairwise comparisons.

- So, when the decision maker is asked to conduct a pair-wise comparison, we must check whether or not comparisons are consistent throughout all attributes/alternatives.
**Analytical Hierarchy process**

**Check Consistency**

- **Transitivity**
  - IF A is more important than B and B is more important than C
  - THEN A is more important than C

- **Cardinal Consistency**
  - IF A is twice as preferable as B and B is three times as preferable as C
  - THEN A must be six times as preferable as C

- In practice, and as the number of alternatives/attributes increases, perfect consistency is very difficult to achieve.
- We check for inconsistency by calculating the matrix maximum Eigen value.

**Analytical Hierarchy process**

**Check Consistency**

- The matrix maximum Eigenvalue $\lambda_{max}$ $\geq n$
- A matrix with perfect consistency occurs when $\lambda_{max} = n$
- The consistency index (CI) is a measure of deviation from perfect consistency: $CI = (\lambda_{max} - n) / (n-1)$
- Perfect consistency occurs when $CI=0$
- CI of a randomly generated matrix is called the Random Index (RI)
- Generally, for acceptable consistency the consistency ratio $CR = CI/RI$ must be less than 10%
Analytical Hierarchy process

Check Consistency

- Step 1: Multiply the pairwise comparison matrix \([A]\) by the relative priority vector \([W]\) to calculate the weighted sum vector \([W']\)

- Step 2: Divide the elements of the vector of weighted sums \([W']\) obtained in Step 1 by the corresponding priority value \([W]\) to calculate the consistency vector \([W'']\)

- Step 3: Compute the average of the values computed in step 2, this average is the maximum Eigen value denoted as \(\lambda_{max}\)

Analytical Hierarchy process

Check Consistency

- Step 4: Compute the consistency index (CI):

\[
CID = \frac{\lambda_{max} - n}{n - 1}
\]

Where \(n\) is the number of items being compared

- Step 5: Compute the consistency ratio (CR):

\[
CR = \frac{CI}{RI}
\]

Where \(RI\) is a random index which is the consistency index of a randomly generated pairwise comparison matrix. \(RI\) depends on the number of elements being compared and takes on the following values.
Analytical Hierarchy process

Check Consistency

- Random index (RI) is the consistency index of a randomly generated pairwise comparison matrix.
- RI depends on the number of elements being compared (i.e., size of pairwise comparison matrix) and takes on the following values:

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>RI</td>
<td>0.00</td>
<td>0.00</td>
<td>0.58</td>
<td>0.90</td>
<td>1.12</td>
<td>1.24</td>
<td>1.32</td>
<td>1.41</td>
<td>1.45</td>
<td>1.49</td>
</tr>
</tbody>
</table>

Inconsistency: Example

- Preferences: If, A ⇒ B (2); B ⇒ C (6)

  Then, A ⇒ C (should be 12) (actually 8)

- Inconsistency

<table>
<thead>
<tr>
<th>Comfort</th>
<th>Car A</th>
<th>Car B</th>
<th>Car C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Car A</td>
<td>1</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>Car B</td>
<td>1/2</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>Car C</td>
<td>1/8</td>
<td>1/6</td>
<td>1</td>
</tr>
</tbody>
</table>
**Analytical Hierarchy process**

### Check Consistency: Example

- **Pair wise comparison matrix**

  \[
  [A] = \begin{bmatrix}
  1 & 2 & 8 \\
  1/2 & 1 & 6 \\
  1/8 & 1/6 & 1 \\
  \end{bmatrix}
  \]

- **Pair wise comparison matrix**

<table>
<thead>
<tr>
<th>Comfort</th>
<th>Car A</th>
<th>Car B</th>
<th>Car C</th>
<th>Row Avg.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Car A</td>
<td>0.615</td>
<td>0.632</td>
<td>0.533</td>
<td>0.593</td>
</tr>
<tr>
<td>Car B</td>
<td>0.308</td>
<td>0.316</td>
<td>0.400</td>
<td>0.341</td>
</tr>
<tr>
<td>Car C</td>
<td>0.077</td>
<td>0.053</td>
<td>0.067</td>
<td>0.066</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
  Column totals: 13/8, 19/6, 15

- **Divide the elements of the vector of weighted sums by the corresponding priority value.**

  \[
  \begin{bmatrix}
  0.593 \\
  0.341 \\
  0.066 \\
  \end{bmatrix}
  \]

  \[
  \begin{bmatrix}
  1.803 \\
  1.034 \\
  0.197 \\
  \end{bmatrix}
  \]

  \[
  \begin{bmatrix}
  3.040 \\
  3.032 \\
  2.985 \\
  \end{bmatrix}
  \]
Analytical Hierarchy process

Check Consistency: Example

- Compute the average of the values computed in step 2 ($\lambda_{max}$)
  \[ \lambda_{max} = \frac{3.040 + 3.032 + 2.985}{3} = 3.019 \]

- Compute the consistency index (CI)
  \[ CI = \frac{\lambda_{max} - n}{n - 1} = \frac{3.019 - 3}{3 - 1} = 0.010 \]

- Compute the consistency ratio (CR)
  \[ CR = \frac{CI}{RI} = \frac{0.010}{0.58} = 0.017 \leq 0.10 \]

- The degree of consistency is acceptable

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Analytical Hierarchy process

Priority Ranking

- The overall priority for each decision alternative is obtained by summing the product of the criterion priority (i.e., weight) (with respect to the overall goal) times the priority (i.e., preference) of the decision alternative with respect to that criterion

- Ranking these priority values, we will have AHP ranking of the decision alternatives
Analytical Hierarchy process

Example

- Find the best alternative (Car A or B or C) considering the selection criteria (price, miles per gallon, comfort, style)

Overall Goal: Select the Best Car

Criteria: Price, MPG, Comfort, Style

Decision Alternatives:

<table>
<thead>
<tr>
<th></th>
<th>Car A</th>
<th>Car B</th>
<th>Car C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MPG</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Comfort</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Style</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Criteria pair-wise comparison matrix

<table>
<thead>
<tr>
<th>Criterion</th>
<th>Price</th>
<th>MPG</th>
<th>Comfort</th>
<th>Style</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>MPG</td>
<td>1/3</td>
<td>1</td>
<td>1/4</td>
<td>1/4</td>
</tr>
<tr>
<td>Comfort</td>
<td>1/2</td>
<td>4</td>
<td>1</td>
<td>1/2</td>
</tr>
<tr>
<td>Style</td>
<td>1/2</td>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Criteria ranking

<table>
<thead>
<tr>
<th>Criterion</th>
<th>Price</th>
<th>MPG</th>
<th>Comfort</th>
<th>Style</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price</td>
<td>0.398</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MPG</td>
<td>0.085</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Comfort</td>
<td>0.218</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Style</td>
<td>0.299</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
**Analytical Hierarchy process**

**Example**

- Alternative pair-wise comparison matrices against each criteria

<table>
<thead>
<tr>
<th>Comfort</th>
<th>Car A</th>
<th>Car B</th>
<th>Car C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Car A</td>
<td>1</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>Car B</td>
<td>1/2</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>Car C</td>
<td>1/8</td>
<td>1/6</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Price</th>
<th>Car A</th>
<th>Car B</th>
<th>Car C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Car A</td>
<td>1</td>
<td>1/3</td>
<td>¼</td>
</tr>
<tr>
<td>Car B</td>
<td>3</td>
<td>1</td>
<td>½</td>
</tr>
<tr>
<td>Car C</td>
<td>4</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>MPG</th>
<th>Car A</th>
<th>Car B</th>
<th>Car C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Car A</td>
<td>1</td>
<td>1/4</td>
<td>1/6</td>
</tr>
<tr>
<td>Car B</td>
<td>4</td>
<td>1</td>
<td>1/3</td>
</tr>
<tr>
<td>Car C</td>
<td>6</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Style</th>
<th>Car A</th>
<th>Car B</th>
<th>Car C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Car A</td>
<td>1</td>
<td>1/3</td>
<td>4</td>
</tr>
<tr>
<td>Car B</td>
<td>3</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>Car C</td>
<td>1/4</td>
<td>1/7</td>
<td>1</td>
</tr>
</tbody>
</table>

**Ranking of each criterion with respect to different alternatives**

<table>
<thead>
<tr>
<th></th>
<th>Price</th>
<th>MPG</th>
<th>Comfort</th>
<th>Style</th>
</tr>
</thead>
<tbody>
<tr>
<td>Car A</td>
<td>0.123</td>
<td>0.087</td>
<td>0.593</td>
<td>0.265</td>
</tr>
<tr>
<td>Car B</td>
<td>0.320</td>
<td>0.274</td>
<td>0.341</td>
<td>0.655</td>
</tr>
<tr>
<td>Car C</td>
<td>0.557</td>
<td>0.639</td>
<td>0.066</td>
<td>0.080</td>
</tr>
</tbody>
</table>

- Sum the product of the criterion priority (with respect to the overall goal) times the priority of the decision alternative with respect to that criterion
### Analytical Hierarchy process

**Example**

- **Ranking of each criterion with respect to different alternatives**

<table>
<thead>
<tr>
<th></th>
<th>Price</th>
<th>MPG</th>
<th>Comfort</th>
<th>Style</th>
</tr>
</thead>
<tbody>
<tr>
<td>Car A</td>
<td>0.123</td>
<td>0.087</td>
<td>0.593</td>
<td>0.265</td>
</tr>
<tr>
<td>Car B</td>
<td>0.320</td>
<td>0.274</td>
<td>0.341</td>
<td>0.655</td>
</tr>
<tr>
<td>Car C</td>
<td>0.557</td>
<td>0.639</td>
<td>0.066</td>
<td>0.080</td>
</tr>
</tbody>
</table>

- **Rank the priority values**

<table>
<thead>
<tr>
<th>Alternative</th>
<th>Priority</th>
</tr>
</thead>
<tbody>
<tr>
<td>Car B</td>
<td>0.421</td>
</tr>
<tr>
<td>Car C</td>
<td>0.314</td>
</tr>
<tr>
<td>Car A</td>
<td>0.265</td>
</tr>
<tr>
<td>Total</td>
<td>1.000</td>
</tr>
</tbody>
</table>