CHAPTER 3

THE SIMPLEX METHOD

In this chapter, Students will be introduced to solve linear programming models using the simplex method. This will give them insights into what commercial linear programming software packages actually do. Such an understanding can be useful in several ways. For example, students will be able to identify when a problem has alternate optimal solutions, unbounded solution, etc.

3.1 Gauss-Jordan Elimination for Solving Linear Equations

The Gauss-Jordan elimination procedure is a systematic method for solving systems of linear equations. It works one variable at a time, eliminating it in all rows but one, and then moves on to the next variable. We illustrate the procedure on three examples.

Example 3.1

\[ \begin{align*}
    x_1 + 2x_2 + x_3 &= 4 \quad (1) \\
    2x_1 - x_2 + 3x_3 &= 3 \quad (2) \\
    x_1 + x_2 - x_3 &= 3 \quad (3)
\end{align*} \]

In the first step of the procedure, we use the first equation to eliminate \( x_1 \) from the other two. Specifically, in order to eliminate \( x_1 \) from the second equation, we multiply the first equation by 2 and subtract the result from the second equation. Similarly, to eliminate \( x_1 \) from the third equation, we subtract the first equation from the third. Such steps are called \textit{elementary row operations}. We keep the first equation and the modified second and third equations. The resulting equations are:

\[ \begin{align*}
    x_1 + 2x_2 + x_3 &= 4 \quad (1) \\
    - 5x_2 + x_3 &= -5 \quad (2) \\
    - x_2 - 2x_3 &= -1 \quad (3)
\end{align*} \]
Note that only one equation was used to eliminate $x_1$ in all the others. This guarantees that the new system of equations has exactly the same solution(s) as the original one. In the second step of the procedure, we divide the second equation by $-5$ to make the coefficient of $x_2$ equal to 1. Then, we use this equation to eliminate $x_2$ from equations 1 and 3. This yields the following new system of equations:

\[
\begin{align*}
    x_1 + \frac{7}{5}x_3 &= 2 \\
    x_2 - \frac{1}{5}x_3 &= 1 \\
    -\frac{11}{5}x_3 &= 0
\end{align*}
\]

Once again, only one equation was used to eliminate $x_2$ in all the others and that guarantees that the new system has the same solution(s) as the original one. Finally, in the last step of the procedure, we use equation 3 to eliminate $x_3$ in equations 1 and 2.

\[
\begin{align*}
    x_1 &= 2 \\
    x_2 &= 1 \\
    x_3 &= 0
\end{align*}
\]

So, there is a unique solution. Note that, throughout the procedure, we were careful to keep three equations that have the same solution(s) as the original three equations. Why is it useful? Because, linear systems of equations do not always have a unique solution and it is important to identify such situations.

**Example 3.2**

\[
\begin{align*}
    x_1 + 2x_2 + x_3 &= 4 \\
    x_1 + x_2 + 2x_3 &= 1 \\
    2x_1 + 3x_2 + 3x_3 &= 2
\end{align*}
\]

First we eliminate $x_1$ from equations 2 and 3.

\[
\begin{align*}
    x_1 + 2x_2 + x_3 &= 4 \\
    -x_2 + x_3 &= -3 \\
    -x_2 + x_3 &= -6
\end{align*}
\]

Then we eliminate $x_2$ from equations 1 and 3.
\[
\begin{align*}
x_1 + 3x_3 &= -2 \quad \text{(1)} \\
x_2 - x_3 &= 3 \quad \text{(2)} \\
0 &= -3 \quad \text{(3)}
\end{align*}
\]

Equation 3 shows that the linear system has no solution.

Example 3.3
\[
\begin{align*}
x_1 + 2x_2 + x_3 &= 4 \quad \text{(1)} \\
x_1 + x_2 + 2x_3 &= 1 \quad \text{(2)} \\
2x_1 + 3x_2 + 3x_3 &= 5 \quad \text{(3)}
\end{align*}
\]

Doing the same as above, we end up with
\[
\begin{align*}
x_1 + 3x_3 &= -2 \quad \text{(1)} \\
x_2 - x_3 &= 3 \quad \text{(2)} \\
0 &= 0 \quad \text{(3)}
\end{align*}
\]

Now equation 3 is an obvious equality. It can be discarded to obtain
\[
\begin{align*}
x_1 &= -2 - 3x_3 \quad \text{(1)} \\
x_2 &= 3 + x_3 \quad \text{(2)}
\end{align*}
\]

The situation where we can express some of the variables (here \(x_1\) and \(x_2\)) in terms of the remaining variables (here \(x_3\)) is important. These variables are said to be basic and non-basic respectively. Any choice of the non-basic variable \(x_3\) yields a solution of the linear system. Therefore the system has infinitely many solutions.

It is generally true that a system of \(m\) linear equations in \(n\) variables has either:

(a) No solution,

(b) A unique solution,

(c) Infinitely many solutions.
The Gauss-Jordan elimination procedure solves the system of linear equations using two elementary row operations:

- Modify some equation by multiplying it by a nonzero scalar (a scalar is an actual real number, such as \( \frac{1}{2} \) or -2; it cannot be one of the variables in the problem),
- Modify some equation by adding to it a scalar multiplied by another equation.

The resulting system of \( m \) linear equations has the same solution(s) as the original system. If an equation \( 0 = 0 \) is produced, it is discarded and the procedure is continued. If an equation \( 0 = a \) is produced where \( a \) is a nonzero scalar, the procedure is stopped: in this case, the system has no solution. At each step of the procedure, a new variable is made basic: it has coefficient 1 in one of the equations and 0 in all the others.

The procedure stops when each equation has a basic variable associated with it. Say \( p \) equations remain (remember that some of the original \( m \) equations may have been discarded). When \( m = p \), the system has a unique solution. When \( m > p \), then \( p \) variables are basic and the remaining \( m - p \) are non-basic. In this case, the system has infinitely many solutions.

### 3.2 The Essence of the Simplex Method

Let’s recall the Example of Section 2.3 of the previous chapter. The graph model of that example is shown in Fig. 3.1. The Five constraints boundaries and their points of intersection are highlighted in the figure. The points of intersection are the corner-point solutions of the problem. The four corner points are (0, 0), (0, 7.5), (1, 7), and (6, 0), are the corner-point feasible solutions (CPF solutions). The other five corner points, (10, 0), (15, 0), (0, 8), (0, 10), and (5, 5.), are called corner-point infeasible solutions.

Properties of the CPF solutions
- If there is exactly one optimal solution, then it must be a CPF solution.
- If there are multiple optimal solutions, then at least two must be adjacent CPF feasible solutions.
- There are only a finite number of CPF solutions.
- If a CPF solution has no adjacent CPF solution that are better as measured by the objective function, then there are no better CPF solutions anywhere; i.e., it is optimal.

In this example, each corner-point solution lies at the intersection of two constraint boundaries (Table 3.1). For any linear programming problem with \( n \) decision variables, two CPF solutions are adjacent to each other if they share \( n-1 \) constraint boundaries. In the current example since \( n=2 \), then two of its CPF solutions are adjacent if they share one constraint boundary. For example, \((0, 0)\) and \((6, 0)\) are adjacent because they share the \( x_2=0 \) constraint boundary. The feasible region of Fig. 3.1 has four edges and each CPF solution has two adjacent two CPF solutions.

<table>
<thead>
<tr>
<th>CPF Solution</th>
<th>Its adjacent CPF solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>(0, 7.5) and (6, 0)</td>
</tr>
<tr>
<td>(0, 7.5)</td>
<td>(1, 7) and (0, 0)</td>
</tr>
<tr>
<td>(1, 7)</td>
<td>(6, 0), and (0, 7.5)</td>
</tr>
<tr>
<td>(6, 0)</td>
<td>(0, 0) and (1, 7)</td>
</tr>
</tbody>
</table>

Table 3.1: CPF solutions

Figure 3.1: Feasible region of the example of section 2.3
Thus, in any linear programming problem that possesses at least one optimal solution, if a CPF solution has no adjacent CPF solutions that are better (as measured by the objective function), then it must be an optimal solution. The point (1, 7) is an optimal solution because its objective equals 7,750,000 is larger than 7,500,000 for (0, 7.5) and 6,000,000 for (6, 0). This is the test that used by the simplex method to determine when an optimal solution has been reached. The general structure of the simplex method is as follow:

3.3 Setting Up the Simplex Method

Before we start discussing the simplex method, we point out that every linear program can be converted into “standard" form:

\[
\begin{align*}
\text{Max} & \quad c_1x_1 + c_2x_2 + \ldots + c_nx_n \\
\text{subject to} & \quad a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\
& \quad \ldots \ldots \ldots \ldots \\
& \quad a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \\
& \quad x_1 \geq 0; \ldots; x_n \geq 0
\end{align*}
\]

Where the objective is maximized, the constraints are equalities and the variables are all nonnegative. This is done as follows:

- If the problem is min z, convert it to max -z.
- If a constraint is \( a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \leq b_i \), convert it into an equality constraint by adding a nonnegative slack variable \( s_i \). The resulting constraint is \( a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n + s_i = b_i \), where \( s_i \geq 0 \).
- If a constraint is \( a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \geq bi \), convert it into an equality constraint by subtracting a nonnegative surplus variable \( s_i \). The resulting constraint is \( a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n - s_i = bi \), where \( s_i \geq 0 \).
- If some variable \( x_j \) is unrestricted in sign, replace it everywhere in the formulation by \( x'_j - x''_j \) where \( x'_j \geq 0 \) and \( x''_j \geq 0 \).
- The right side of an equation can always be made nonnegative by multiplying both sides by \(-1\). For example, \(-2x_1 + 3x_2 = -5\) is mathematically equivalent to \(+2x_1 - 3x_2 = +5\).
- The direction of an inequality is reversed when both sides are multiplied by \(-1\). For example whereas \( 2 < 4 \), \(-2 > -4\). Thus the inequality \( 2x_1 - x_2 \leq -5 \) can be replaced by \(-2x_1 + x_2 \geq +5\).

**Example 3.4**

Transform the following linear program into standard form.

\[
\begin{align*}
\text{Min} & \quad -2x_1 + 3x_2 \\
& x_1 - 3x_2 + 2x_3 \leq 3 \\
& -x_1 + 2x_2 \quad \geq 2 \\
& x_1 \text{ is unrestricted; } x_2 \geq 0; \, x_3 \geq 0
\end{align*}
\]

Let us first turn the objective into a max and the constraints into equalities.

\[
\begin{align*}
\text{Max} & \quad 2x_1 - 3x_2 \\
& x_1 - 3x_2 + 2x_3 + s_i = 3 \\
& -x_1 + 2x_2 - s_2 = 2 \\
& x_1 \text{ unrestricted; } x_2 \geq 0; \, x_3 \geq 0; \, s_i \geq 0; \, s_2 \geq 0
\end{align*}
\]

The last step is to convert the unrestricted variable \( x_1 \) into two nonnegative variables; where: \( x_1 = x'_1 - x''_1 \). Then, the system of equations is transferred as follow:

\[
\begin{align*}
\text{Max} & \quad 2x'_1 - 2x''_1 - 3x_2 \\
& x'_1 - x''_1 - 3x_2 + 2x_3 + s_i = 3 \\
& -x'_1 + x''_1 + 2x_2 - s_2 = 2 \\
& x'_1 \geq 0; \, x''_1 \geq 0; \, x_2 \geq 0; \, x_3 \geq 0; \, s_i \geq 0; \, s_2 \geq 0
\end{align*}
\]
These slack variables represent the part that is not used or not available from each resource in each constraint. Also, these variables represent the difference between the actual needs and the availability.

This new form of the problem is called the augmented form, because the original form has been augmented by some supplementary variables needed to apply the simplex method. If the slack variable equals zero in the current solution, then this solution lies on the constraint boundary for the corresponding functional constraint. A value greater than zero means that the solution lies on the feasible side of this constraint boundary, whereas a value less than zero means that the solution lies on the infeasible side of this constraint boundary.

An augmented solution is a solution for the original variables (the decision variables) that has been augmented by the corresponding values of the slack variables. A basic solution is an augmented corner-point solution. This can be either feasible or infeasible. Accordingly, a basic feasible (BF) solution is an augmented CPF solution.

3.4 Solution of Linear Programs by the Simplex Method

For simplicity, in this following example we solve the case where the constraints are of the form ≤ and the right-hand-sides are nonnegative. We will explain the steps of the simplex method while we progress through an example.

Example 3.5

\[
\begin{align*}
\text{Max} & \quad x_1 + x_2 & \quad (0) \\
2x_1 + x_2 & \leq 4 & \quad (1) \\
x_1 + 2x_2 & \leq 3 & \quad (2) \\
x_1 & \geq 0; \ x_2 & \geq 0
\end{align*}
\]

First, we convert the problem into standard form by adding slack variables \(s_1, s_2 \geq 0\).

\[
\begin{align*}
\text{Max} & \quad x_1 + x_2 & \quad (0) \\
2x_1 + x_2 + s_1 & = 4 & \quad (1) \\
x_1 + 2x_2 + s_2 & = 3 & \quad (2) \\
x_1 & \geq 0; \ x_2 & \geq 0 \quad s_1 \geq 0; \ s_2 \geq 0
\end{align*}
\]
Let $z$ denote the objective function value. Here, $z = x_1 + x_2$ or, equivalently, $z - x_1 - x_2 = 0$.

Putting this equation together with the constraints, we get the following system of linear equations.

\[
\begin{align*}
z - x_1 - x_2 &= 0 \quad \text{Row (0)} \\
2x_1 + x_2 + s_1 &= 4 \quad \text{Row (1)} \\
x_1 + 2x_2 + s_2 &= 3 \quad \text{Row 2 (2)}
\end{align*}
\]

Our goal is to maximize $z$, while satisfying these equations and, in addition, $x_1 \geq 0$, $x_2 \geq 0$, $s_1 \geq 0$, $s_2 \geq 0$. Note that the equations are already in the form that we expect at the last step of the Gauss-Jordan procedure. Namely, the equations are solved in terms of the non-basic variables $x_1$, $x_2$. The variables (other than the special variable $z$) which appear in only one equation are the basic variables. Here the basic variables are $s_1$ and $s_2$.

Generally, assume that the standard form has $m$ equations and $n$ variables ($m \leq n$). A solution can be obtained when exactly $n - m$ variables are set equal to zero. The unique solutions resulting from setting $n - m$ variables equal to zero are called basic solutions. If a basic solution satisfies the non-negativity restrictions, it is called a feasible basic solution. The variables set equal to zero are called non-basic variables; the remaining ones are called basic variables.

The simplex algorithm starts at the origin, which is usually referred to as the starting solution. It then moves to an adjacent corner point.

**Initialization**

An initial basic feasible solution is obtained from the system of equations by setting the non-basic variables to zero ($x_1 = x_2 = 0$). This solution maximizes the values of the slack variables and this is corresponding to the origin point of the graphical solution. Here this yields $x_1 = x_2 = 0; s_1 = 4; s_2 = 3; z = 0$. This solution is mathematically correct. However, the question raised here: Is this an optimal solution or can we increase $z$ (Our goal)?

**Optimality test**
The previous solution yields $z = 0$. By looking at Row 0 above, we can increase $z$ by increasing $x_1$ or $x_2$. This is because these variables have a negative coefficient in Row 0 and none of the basic variables have a nonzero coefficient in the objective function. If all coefficients in Row 0 had been non-negative, we could have concluded that the current basic feasible solution is optimum, since there would be no way to increase $z$ (remember that all variables must remain $\geq 0$). Thus, the first rule of the simplex method:

**Rule 1**: If all variables have a nonnegative coefficient in Row 0, the current basic feasible solution is optimal. Otherwise, pick a variable with a negative coefficient in Row 0.

**Determining the direction of movement (Step 1)**

The choice of which non basic variable to increase is based on how much this variable influence the objective function, the one that increases the objective function the most will be chosen. Increasing this non-basic variable from zero will convert it to basic variable. The variable chosen by Rule 1 is called the entering variable. Therefore, this variable is called the **entering basic variable**. Here let us choose, say, $x_1$ as our entering variable. It really does not matter which variable we choose as long as it has a negative coefficient in Row 0 and both variables has the same effect on the objective function.

However, we always select the variable with the highest negative coefficient because such selection is more likely to lead to the optimal solution rapidly.

**Determining where to stop (Step 2)**

In this step, it is required to determine how far to increase the entering basic variable $x_1$ before stopping. Increasing $x_1$ increases $z$, so we want to go as far as possible without leaving the feasible region. So, setting $x_2 = 0$, yields the following solution.

$$
\begin{align*}
  x_2 & = 0 \\
  s_1 & = 4 - 2x_1 \\
  s_2 & = 3 - x_1
\end{align*}
$$
How far $x_1$ can be increased without violating the non-negativity constraint for the basic and non-basic variables? Thus, $x_1$ can be increased just to 2, at which point $s_1$ dropped to zero. Increasing $x_1$ beyond 2 would cause $s_1$ to become negative, which would violate feasibility. This is called the **minimum ratio test**, which is the ratio of the right hand side to the coefficient of the entering basic variable (except we ignore any equation where this coefficient is zero or negative, since this coefficient leads to no upper bound on the entering basic variable). Decreasing the basic variable to zero will convert it non-basic variable for the next basic feasible solution. Therefore, this variable is called the **leaving basic variable** because it is leaving the basic variables.

**Solving for the new basic feasible solution (Step 3)**

Increasing $x_1 = 0$ to $x_1 = 2$ moves us from the initial basic feasible solution on the left to the new basic feasible solution on the right.

<table>
<thead>
<tr>
<th>Initial basic feasible solution</th>
<th>New basic feasible solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-basic variables: $x_1 = 0$, $x_2 = 0$</td>
<td>$x_2 = 0$, $s_1 = 0$</td>
</tr>
<tr>
<td>Basic variable: $s_1 = 4$, $s_2 = 3$</td>
<td>$x_1 = 2$, $s_2 = ?$</td>
</tr>
</tbody>
</table>

The purpose of this step is to convert to more convenient form for conducting the optimality test. This form also will be used to identify the value of the $s_2$ for the new solution. Then, solve these equations for the basic variables $x_1$ and $s_2$.

$$
\begin{align*}
    z & - x_1 - x_2 = 0 \quad \text{Row 0} \\
    2x_1 + x_2 + s_1 & = 4 \quad \text{Row 1} \\
    x_1 + 2x_2 + s_2 & = 3 \quad \text{Row 2}
\end{align*}
$$

These yield

$$
\begin{align*}
    z & - \frac{1}{2}x_2 + \frac{1}{2}s_1 = 2 \quad \text{Row 0} \\
    x_1 & + \frac{1}{2}x_2 + \frac{1}{2}s_1 = 2 \quad \text{Row 1} \\
    \frac{3}{2}x_2 - \frac{1}{2}s_1 + s_2 & = 1 \quad \text{Row 2}
\end{align*}
$$
With basic solution \( x_2 = s_1 = 0; \ x_1 = 2; \ s_2 = 1; \ z = 2 \). We have just discovered the second rule of the simplex method.

**Rule 2:** For each Row \( i, i \geq 1 \), where there is a strictly positive “entering variable coefficient”, compute the ratio of the Right Hand Side to the “entering variable coefficient”. Choose the pivot row as being the one with Minimum ratio.

Once you have identified the pivot element by Rule 2, a Gauss-Jordan pivot is performed. This gives a new basic solution. Is it an optimal solution? This question is addressed by Rule 1, so we have closed the loop. The simplex method iterates between Rules 1, 2 and pivoting until Rule 1 guarantees that the current basic solution is optimal. That's all there is to the simplex method.

**Optimality test for the new basic feasible solution**

Increasing the non-basic variables of \( z = 2 + 1/2x_2 - 1/2s_1 \), from zero would result in moving toward one of the adjacent basic feasible solutions. Because \( x_2 \) has positive coefficient, increasing \( x_2 \) would lead to adjacent basic feasible solution that is better than the current basic feasible solution, so the current solution is not optimal. Where should we pivot? Rule 2 tells us to pivot in Row 2, since the ratios are \( 2/(1/2) = 4 \) for Row 1, and \( 1/(3/2) = 2/3 \) for Row 2, and the minimum occurs in Row 2. So we pivot on \( 3/2x_2 \) in the above system of equations. The leaving basic variable is \( s_2 \). These yields:

\[
\begin{align*}
\text{Row 0} & : \ z + 1/3s_1 + 1/3s_2 = 7/3 \\
\text{Row 1} & : \ x_1 + 2/3s_1 - 1/3s_2 = 5/3 \\
\text{Row 2} & : \ x_2 - 1/3s_1 + 2/3s_2 = 2/3
\end{align*}
\]

With basic solution \( s_1 = s_2 = 0; \ x_1 = 5/3; \ x_2 = 2/3; \ z = 7/3 \). Now Rule 1 tells us that this basic solution is optimal, since there are no more negative entries in Row 0. All the above computations can be represented very compactly in tableau form (Table 3.2).
### Table 3.2: Simplex tableau of Example 3.5

<table>
<thead>
<tr>
<th>Basic variables</th>
<th>Coefficient of</th>
<th>Right-hand side (solution)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( z )</td>
<td>( x_1 )</td>
</tr>
<tr>
<td>( z )</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( s_1 )</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( z )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( z )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

It is interesting to interpret the steps of the simplex method graphically (Fig. 3.2). The simplex method starts at the corner point \((x_1 = 0; \ x_2 = 0)\) with \(z = 0\). Then it discovers that \(z\) can increase by increasing, say, \(x_1\). Since we keep \(x_2 = 0\), this means we move along the \(x_1\) axis. How far can we go? Only until we hit a constraint; otherwise, the solution would become infeasible. That's exactly what Rule 2 of the simplex method does. The minimum ratio rule identifies the first constraint that will be encountered. And when the constraint is reached, its slack \(s_1\) becomes zero. So, after the first pivot, we are at point B \((x_1 = 2; \ x_2 = 0)\). Rule 1 discovers that \(z\) can be increased by increasing \(x_2\) while keeping \(s_1 = 0\). This means that we move along the boundary of the feasible region \(2x_1 + x_2 = 4\) until we reach another constraint. After pivoting, we reach the optimal point D \((x_1 = 5/3; \ x_2 = 2/3)\).
3.5 The Simplex Method in Tabular Format

The tabular form of the simplex method is mathematically equivalent to the algebraic form. However, instead of writing down each set of equations in full detail, we use a **simplex tableau** to record only the essential information, namely: the coefficients of the variables; the constants on the right-hand side of the equations; and the basic variables appearing in each equation. To introduce the tabular form, let’s consider the augmented form of Example 3.5:

\[
\begin{align*}
 z - x_1 - x_2 &= 0 \\
 2x_1 + x_2 + s_1 &= 4 \\
 x_1 + 2x_2 + s_2 &= 3 \\
 x_1 \geq 0; x_2 \geq 0; s_1 \geq 0; s_2 \geq 0
\end{align*}
\]

This system of equations can be expressed as shown in Table 3.3.

<table>
<thead>
<tr>
<th>Basic variables</th>
<th>Coefficient of</th>
<th>Right-hand side (solution)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$z$ $x_1$ $x_2$ $s_1$ $s_2$</td>
<td></td>
</tr>
<tr>
<td>$z$</td>
<td>1   1  -1  0  0</td>
<td>0</td>
</tr>
<tr>
<td>$s_1$</td>
<td>0   2  1  1  0</td>
<td>4; $4/2=2$</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0   1  2  0  1</td>
<td>3; $3/1=3$</td>
</tr>
</tbody>
</table>

The column on the left indicates which basic variable appears in each equation for the current basic feasible solution. After setting $x_1 = 0$ and $x_2 = 0$, the right-side column gives the resulting solution for basic variables, so that the initial basic feasible solution is $x_1 = x_2 = 0; s_1 = 4; s_2 = 3$ with $z = 0$. Notice that, the columns for the basic variables always contains only one nonzero coefficient and this coefficient is 1 in the row of this basic variable. So, this basic variable always equals the constant on the right-hand side of its equation.

The current basic feasible solution is optimal if and only if every coefficient in the $z$ equation is non-negative ($\geq 0$). If it is, then stop; otherwise, go to the next step, which changing one non-basic variable to a basic variable, and one basic variable to a non-basic variable and then solving for the new set of equations.
**Determine the entering basic variable** by selecting the variable with the negative coefficient having the largest absolute value in the z equation (first row in the previous table). Let us choose \( x_1 \) as both variable have the same coefficient. Call this column that contains this variable as the **pivot column**.

**Determine the leaving basic variable** by dividing the right-hand side by the positive (non-zero) coefficients only of the pivot column. Select the smallest one, the basic variable for this equation will be the leaving basic variable. The row that contains the smallest value is called the **pivot row**. The number that lies on both the pivot row and the pivot column called the **pivot number**.

Determine the new basic feasible solution by constructing a new simplex tableau in proper form from Gaussian elimination. Change the coefficient of the new basic variable in the pivot row to one by dividing the entire row by the pivot number. We need to continue to obtain a coefficient of zero for the new basic variable \( x_1 \) in the other rows (Table 3.4) The first and third rows have coefficients of -1 and 1 for \( x_1 \), so each of these rows needs to be changed by using the following formula.

\[
\text{New row} = \text{old row} - (\text{pivot column coefficient} \times \text{new pivot row})
\]

<table>
<thead>
<tr>
<th>Old row (z)</th>
<th>(1  -1   -1   0  0  0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pivot column coefficient: (-1) x</td>
<td>(0  1  (\frac{1}{2})  (\frac{1}{2})  0  2)</td>
</tr>
<tr>
<td>New row</td>
<td>1  0  -(\frac{1}{2})  (\frac{1}{2})  0  2)</td>
</tr>
</tbody>
</table>

Table 3.4: Simplex tableau after first pivoting

<table>
<thead>
<tr>
<th>Basic variables</th>
<th>Coefficient of</th>
<th>Right-hand side (solution)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( z ) ( x_1 ) ( x_2 ) ( s_1 ) ( s_2 )</td>
<td></td>
</tr>
<tr>
<td>( z )</td>
<td>1  -1</td>
<td>0  0  0</td>
</tr>
<tr>
<td>( s_1 )</td>
<td>0  2</td>
<td>1  1  0 (=4)</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>0  1</td>
<td>2  0  1 (=3)</td>
</tr>
<tr>
<td>( z )</td>
<td>1  0 (=\frac{1}{2})</td>
<td>1/2 0 2</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>0 1 (=\frac{1}{2})</td>
<td>(\frac{1}{2}) 0 (=2)</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>0 0 (=\frac{3}{2})</td>
<td>-1/2 1 (=2/3)</td>
</tr>
</tbody>
</table>
The right-side column gives the resulting solution for basic variables, so that the basic feasible solution is \( x_1 = 2; \ x_2 = 0; \ s_1 = 0; \ s_2 = 1 \) with \( z = 2 \).

The next iteration, we start from the new tableau. The entering basic variable is \( x_2 \) as it has the negative coefficient in the \( z \) function. From the minimum ratio test, \( s_2 \) is the leaving basic variable. The pivot column is the \( x_2 \) column and then pivot row is the \( s_2 \) row and the 3/2 value is the pivot number (Table 3.5).

Table 3.5: Complete set of the simplex tableau

<table>
<thead>
<tr>
<th>Basic variables</th>
<th>Coefficient of</th>
<th>Right-hand side (solution)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( z )</td>
<td>( x_1 )</td>
</tr>
<tr>
<td>( z )</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( s_1 )</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( z )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( z )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

### 3.6 Special Cases

#### 3.6.1 Tie for the Entering Basic Variables

Suppose that there are two or more non-basic variables having the same largest negative coefficients. This would occur if the objective function has the following form: \( z = x_1 + x_2 \). So, how this tie could be broken? Consider the same example discussed before in Section 3.5 as shown below (Table 3.6). In this case, the selection between these variables may be made arbitrarily as the optimal solution will be reached eventually regardless of the variable chosen.

\[
\begin{align*}
\text{max} \quad & x_1 + x_2 \\
\text{subject to} \quad & 2x_1 + x_2 \leq 4 \\
& x_1 + 2x_2 \leq 3 \\
& x_1 \geq 0; \ x_2 \geq 0
\end{align*}
\]
Standard form \[ z - x_1 - x_2 = 0 \]
\[ 2x_1 + x_2 + s_1 = 4 \]
\[ x_1 + 2x_2 + s_2 = 3 \]
\[ x_1 \geq 0; \ x_2 \geq 0; \ s_1 \geq 0; \ s_2 \geq 0 \]

Table 3.6: Tie of entering basic variables

<table>
<thead>
<tr>
<th>Basic variables</th>
<th>Coefficient of</th>
<th>Right-hand side (solution)</th>
</tr>
</thead>
<tbody>
<tr>
<td>z</td>
<td>1 -1 -1 0 0 0</td>
<td>0</td>
</tr>
<tr>
<td>s_1</td>
<td>0 2 1 1 0 4</td>
<td>4; 4/2=2</td>
</tr>
<tr>
<td>s_2</td>
<td>0 1 2 0 1 3</td>
<td>3; 3/1=3</td>
</tr>
</tbody>
</table>

3.6.2 Alternate Optimal Solutions

Let us solve a small variation of the earlier example, with the same constraints but a slightly different objective:

\[ \text{max} \quad x_1 + 1/2x_2 \]
\[ 2x_1 + x_2 \leq 4 \]
\[ x_1 + 2x_2 \leq 3 \]
\[ x_1 \geq 0; \ x_2 \geq 0 \]

As before, we add slacks \( s_1 \) and \( s_2 \), and we solve by the simplex method, using tableau representation (Table 3.7).

Table 3.7: Alternate optimal solutions

<table>
<thead>
<tr>
<th>Basic variables</th>
<th>Coefficient of</th>
<th>Right-hand side (solution)</th>
</tr>
</thead>
<tbody>
<tr>
<td>z</td>
<td>1 -1 -1/2 0 0 0</td>
<td>0</td>
</tr>
<tr>
<td>s_1</td>
<td>0 2 1 1 0 4</td>
<td>4</td>
</tr>
<tr>
<td>s_2</td>
<td>0 1 2 0 1 3</td>
<td>3</td>
</tr>
<tr>
<td>z</td>
<td>1 0 0 1/2 0 2</td>
<td>2</td>
</tr>
<tr>
<td>x_1</td>
<td>0 1 1/2 1/2 0 2</td>
<td>2; 2/(1/2) = 4</td>
</tr>
<tr>
<td>s_2</td>
<td>0 0 3/2 -1/2 1 1</td>
<td>1; 1/(3/2) = 2/3</td>
</tr>
</tbody>
</table>
Now Rule 1 shows that this is an optimal solution. Interestingly, the coefficient of the non-basic variable $x_2$ in Row 0 (objective function row) equals zero. Going back to the rationale that allowed us to derive Rule 1, we observe that, if we increase $x_2$ (from its current value of 0), this will not affect the value of $z$. Increasing $x_2$ produces changes in the other variables, of course, through the equations in Rows 1 and 2 (constraint 1 and 2 rows). In fact, we can use Rule 2 and pivot to get a different basic solution with the same objective value $z = 2$ (Table 3.8).

<table>
<thead>
<tr>
<th>Basic variables</th>
<th>Coefficient of</th>
<th>Right-hand side (solution)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z$</td>
<td>$1$ $0$ $0$ $\frac{1}{2}$ $0$</td>
<td>2</td>
</tr>
<tr>
<td>$x_1$</td>
<td>$0$ $1$ $0$ $\frac{2}{3}$ $\frac{-1}{3}$</td>
<td>$\frac{5}{3}$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$0$ $0$ $1$ $\frac{-1}{3}$ $\frac{2}{3}$</td>
<td>$\frac{2}{3}$</td>
</tr>
</tbody>
</table>

Note that the coefficient of the non-basic variable $s_2$ in Row 0 is equal to 0. Using $s_2$ as entering variable and pivoting, we would recover the previous solution.

So, whenever a problem has more than one optimal basic feasible solution, at least one of the non-basic variables has a coefficient of zero in the final $z$ equation. Accordingly, increasing any such variable would not change the value of $Z$. Therefore, these other optimal basic feasible solutions can be identified by performing additional iterations of the simplex method. Each time choose a non-basic variable with a zero coefficient as the entering basic variable.

3.6.3 Degeneracy (Tie for the Leaving Basic Variable)

Example 3.6

$$\text{max} \quad 2x_1 + x_2$$
$$3x_1 + x_2 \leq 6$$
$$x_1 - x_2 \leq 2$$
$$x_2 \leq 3$$
$$x_1 \geq 0; \ x_2 \geq 0$$
Let us solve this problem using the simplex method. In the initial tableau, we can choose \( x_1 \) as the entering variable (Rule 1) and Row 2 as the pivot row (the minimum ratio in Rule 2 is a tie, and ties are broken arbitrarily). We pivot and this yields the tableau below (Table 3.9).

<table>
<thead>
<tr>
<th>Basic variables</th>
<th>Coefficient of</th>
<th>Right-hand side (solution)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z )</td>
<td>1 -2 -1 0 0 0 0</td>
<td></td>
</tr>
<tr>
<td>( s_1 )</td>
<td>0 3 1 1 0 0</td>
<td>6; 6/3 = 2</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>0 1 -1 0 1 0</td>
<td>2; 2/1 = 2</td>
</tr>
<tr>
<td>( s_3 )</td>
<td>0 0 0 0 1 1</td>
<td>3</td>
</tr>
</tbody>
</table>

Note that this basic solution has a basic variable \( (s_1) \) equals zero. The variables that did not chosen to be leaving variables will have a value of zero in the new feasible solution. Basic variables with a value of zero are called degenerate, and the same term is applied to same corresponding feasible solution. When this occurs, we say that the basic solution is degenerate. Let us continue the steps of the simplex method. Rule 1 indicates that \( x_2 \) is the entering variable. Now let us apply Rule 2. The ratios to consider are 0/4 in Row 1 and 3/1 in Row 3. The minimum ratio occurs in Row 1, so let us perform the corresponding pivot (Table 3.10).

<table>
<thead>
<tr>
<th>Basic variables</th>
<th>Coefficient of</th>
<th>Right-hand side (solution)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z )</td>
<td>1 0 -3 0 2 0</td>
<td>4</td>
</tr>
<tr>
<td>( s_1 )</td>
<td>0 0 4 1 -3 0</td>
<td>0; 0/4 = 0</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>0 1 -1 0 1 0</td>
<td>2;</td>
</tr>
<tr>
<td>( s_3 )</td>
<td>0 0 1 0 0 1</td>
<td>3; 3/1 = 3</td>
</tr>
</tbody>
</table>

We get exactly the same solution! The only difference is that we have interchanged the names of a non-basic variable with that of a degenerate basic variable \( (x_2 \) and \( s_1 \)). Rule 1 tells us the solution is not optimal, so let us continue the steps of the simplex method.
Variable $s_2$ is the entering variable and the last row wins the minimum ratio test. After pivoting, we get the tableau (Table 3.11):

Table 3.11: Final simplex tableau of Example 3.6

<table>
<thead>
<tr>
<th>Basic variables</th>
<th>Coefficient of Right-hand side</th>
<th>$z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z$</td>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2/3</td>
<td>0</td>
<td>1/3</td>
</tr>
<tr>
<td>$x_2$</td>
<td></td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$x_1$</td>
<td></td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1/3</td>
<td>0</td>
<td>-1/3</td>
</tr>
<tr>
<td>$s_2$</td>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-1/3</td>
<td>1</td>
<td>4/3</td>
</tr>
</tbody>
</table>

By Rule 1, this is the optimal solution. So, after all, degeneracy did not prevent the simplex method to find the optimal solution in this example. It just slowed things down a little. Unfortunately, on other examples, degeneracy may lead to cycling, i.e. a sequence of pivots that goes through the same tableaus and repeats itself indefinitely. In theory, **cycling can be avoided by choosing the entering variable with smallest index in Rule 1**, among all those with a negative coefficient in Row 0, and by breaking ties in the minimum ratio test by choosing the leaving variable with smallest index. This rule, although it guaranties that cycling will never occur, turns out to be somewhat inefficient. Actually, in commercial codes, no effort is made to avoid cycling. This may come as a surprise, since degeneracy is a frequent occurrence. But there are two reasons for this:

- Although degeneracy is frequent, cycling is extremely rare.
- The precision of computer arithmetic takes care of cycling by itself: round off errors accumulate and eventually gets the method out of cycling

### 3.6.4 Unbounded Optimum (No leaving basic variable)

**Example 3.7**

$$\text{Max} \quad 2x_1 + x_2$$

- $x_1 + x_2 \leq 1$
- $x_1 - 2x_2 \leq 2$
- $x_1 \geq 0; \quad x_2 \geq 0$
Solving by the simplex method, we get the following tableau. At the second stage, Rule 1 chooses $x_2$ as the entering variable, but there is no ratio to compute, since there is no positive entry in the column of $x_2$. As we start increasing $x_2$, the value of $z$ increases (from Row 0) and the values of the basic variables increase as well (from Rows 1 and 2). There is nothing to stop them going off to infinity. So the problem is unbounded (Table 3.12).

Table 3.12: Simplex tableau of Example 3.7

<table>
<thead>
<tr>
<th>Basic variables</th>
<th>Coefficient of</th>
<th>Right-hand side (solution)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$z$</td>
<td>$x_1$</td>
</tr>
<tr>
<td>-----------------</td>
<td>-----</td>
<td>-------</td>
</tr>
<tr>
<td>$z$</td>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>$s_1$</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$z$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$s_1$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

### 3.6.5 Infeasible solution

Referring to Fig. 3.2, we may notice that not all extreme points are feasible solutions. Only those points form the boundary of the feasible region are feasible. An infeasible extreme point might provide a better value for the objective function, yet would not be a practical solution. The extreme points C (3, 0) and F (0, 4) of Fig. 3.2 are infeasible as at each point, at least one constraint is violated in the original problem. How can we identify these infeasible solutions without the aid of a graphical solution space?

All extreme points of Fig. 3.2 are listed in the Table (3.13). Each row corresponds to a unique solution labeled in Fig. 3.2, and the entry in each column shows the values for the corresponding variable at that solution. It is clear that for each solution exactly two variables are non-basic (difference between number of variables, four, and equation, two) with zero values at all solutions. Infeasible solutions where at least one variable has a negative value are shaded as shown in Table (3.13).
Table 3.13: Variables and their values at all extreme points of Example 3.5

<table>
<thead>
<tr>
<th>Point</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>D</td>
<td>5/3</td>
<td>2/3</td>
<td>0</td>
<td>0</td>
<td>7/3</td>
</tr>
<tr>
<td>E</td>
<td>0</td>
<td>1.5</td>
<td>2.5</td>
<td>0</td>
<td>1.5</td>
</tr>
<tr>
<td>F</td>
<td>0</td>
<td>4</td>
<td>0</td>
<td>-5</td>
<td>4</td>
</tr>
</tbody>
</table>

It can be seen that infeasible solutions are characterized by the presence of at least one negative valued variable. Accordingly, any solution to a linear program is infeasible if any variable in the augmented form of a given problem is negative at that solution.

3.6.6 Equality Constraints

Any equality can be substituted by two inequality constraints. For example:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

is equivalent to:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$$

and

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \geq b_1$$

Dealing with equality constraints will be discussed on the following example.

Example 3.8

Max $3x_1 + 5x_2$

$x_1 \leq 4$

$2x_2 \leq 12$

$3x_1 + 2x_2 = 18$

$x_1 \geq 0; x_2 \geq 0$

The augmented form becomes

$$z - 3x_1 - 5x_2 = 0 \quad (0)$$

$$x_1 + s_1 = 4 \quad (1)$$

$$2x_2 + s_2 = 12 \quad (2)$$

$$3x_1 + 2x_2 = 18 \quad (3)$$

$x_1 \geq 0; x_2 \geq 0; s_1 \geq 0; s_2 \geq 0$
In the above formulation, the problem does not have basic feasible solution as there is no slack variable to be used as the initial basic variable for Eq. (3). Accordingly, a non-negative artificial variable is introduced as slack variable for Eq. (3). Then, Eq. (3) becomes:

$$3x_1 + 2x_2 + s_3 = 18$$  \hspace{1cm} (3)

Now, these equations have an initial basic solution as follow:

$$(x_1, \ x_2, \ s_1, \ s_2, \ s_3) = (0, \ 0, \ 4, \ 12, \ 18)$$

The augmented form of the current problem after adding the artificial variable is shown by the shaded area in the following figure. However, the solution lies on the bold line segment. This is represented mathematically by having $s_3 = 0$. So, the initial basic solution is also feasible for the original problem if we put $s_3 = 0$.

![Figure 3.3: Feasible region of Example 5.3](image)

To bring $s_3$ to zero, a huge number is multiplied by $s_3$ and then subtracted from the objective function. Thus the objective function becomes:

Maximize $z = 3x_1 + 5x_2 - Ms_3$  \hspace{1cm} (0)

The added value "M" is called "the big M method". Then, maximum $z$ occurs only when $s_3 = 0$. When forming the simplex tableau, it looks as follow (Table 3.14):

Table 3.14: Initial simplex tableau of Example 3.8
The form presented in the simplex tableau is not proper to apply Gaussian elimination where each equation should have only one basic variable. Equation (0) has two basic variables $z$ and $s_3$. So, each basic variable should be eliminated from Eq. (0) except the "$z$". To eliminate $s_3$ from Eq. (0), then subtract Eq. (3) from Eq. (0) as follow:

$$z - (3 + 3M) x_1 - (5 + 2M) x_2 = -18M$$

Finally, the simplex tableau is formulated as follow (Table 3.15):

<table>
<thead>
<tr>
<th>Basic variables</th>
<th>Coefficient of</th>
<th>Right-hand side (solution)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z$</td>
<td>1</td>
<td>-3M-3 0 0 0 0,0 M-18M</td>
</tr>
<tr>
<td>$s_1$</td>
<td>0</td>
<td>1 0 1 0 0 0,4 4/1=4</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0</td>
<td>0 2 0 1 0 0,12</td>
</tr>
<tr>
<td>$s_3$</td>
<td>0</td>
<td>3 2 0 0 1 0,18 18/3=6</td>
</tr>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>1 0 1 0 0 0 4 4/1=4</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0</td>
<td>0 2 0 1 0 0 12 12/2=6</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>3 2 -3 0 1 0 6 6/3=3</td>
</tr>
<tr>
<td>$z$</td>
<td>1</td>
<td>0 0 -3/2 0 0 0 27 27/1=27</td>
</tr>
<tr>
<td>$x_1$</td>
<td>0</td>
<td>1 0 1 0 0 4 4/1=4</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0</td>
<td>0 0 3 1 -1 6 6/3=2</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>0 1 0 1/2 0 2 2 2/1=2</td>
</tr>
</tbody>
</table>

Then, the variables are $x_1 = 2; x_2 = 6$ with $z = 3 x 2 + 5 x 6 = 36$

3.6.7 The $\geq$ Inequality Constraints
The direction of an inequality always is reversed by multiplying both sides by \((-1)\). Accordingly, any functional constraint of the \(\geq\) form can be converted into the standard conventional constraint \(\leq\) form by changing the signs of all the members of both sides. Let’s consider the following example:

**Example 3.9**

\[
\begin{align*}
\text{Min} & \quad z = 0.4x_1 + 0.5x_2 \\
\text{Subject to} & \quad 0.3x_1 + 0.1x_2 \leq 2.7 \\
& \quad 0.5x_1 + 0.5x_2 = 6 \\
& \quad 0.6x_1 + 0.4x_2 \geq 6 \\
& \quad x_1 \geq 0; \ x_2 \geq 0
\end{align*}
\]

Using this approach, the third constraint of Example 3.9, will read:

\[-0.6x_1 - 0.4x_2 \leq -6\]

Then, changing this inequality to an equation format by adding a surplus variable, the equation will read as follow:

\[-0.6x_1 - 0.4x_2 + s_3 = -6\]

### 3.6.8 Negative Right-hand Sides

The standard form of the simplex method assumes that the right-hand side is always positive. Also, it assumes that all variable values are greater than or equal to zero. As it has been introduced in section 3.6.5, if the right hand side for any equation is negative, then, the resulting basic solution is infeasible. As shown in the previous section, a negative right-hand side, such as in the third constraint of Example 3.9, would give a negative value for the slack variable \(s_3 = -6\), in the initial basic solution (where \(x_1 = 0, \ x_2 = 0\)) which violates the non-negativity constraint for this variable. Multiplying both sides of the equation by \((-1)\), makes the right-hand side positive as shown below:

\[0.6x_1 + 0.4x_2 - s_3 = 6\]

but it changes the sign of the slack variable to \(-1\), so the variable still would be negative. However, in this form, the constraint can be viewed as an equality constraint that needs
an *artificial variable* as discussed before in section 3.6.6. Let $s_4$ be a non-negative artificial variable greater than zero, the third constraint of Example 3.9 becomes:

$$0.6x_1 + 0.4x_2 - s_3 + s_4 = 6$$

In this case $s_4$ is used as the initial basic variable ($s_4 = 6$) for this equation and $s_3$ begins as a non-basic variable. The big M method would also be applied as presented in section 3.6.6 for solving equality constraints. Now, the original constraint have been twice revised by expanding its feasible region, first by introducing the surplus variable $s_3$ to convert it to equality constraint and then by introducing the artificial variable $s_4$. Finally, any $\geq$ could be simply transformed as follow:

$$0.6x_1 + 0.4x_2 \geq 6$$

Change to:

$$0.6x_1 + 0.4x_2 - s_3 = 6 \quad (s_3 \geq 0)$$

Then:

$$0.6x_1 + 0.4x_2 - s_3 + s_4 = 6 \quad (s_3 \geq 0, s_4 \geq 0)$$

In this final form, $s_3$ is called a surplus variable because it subtracts the surplus of the left-hand side over the right-hand side to convert the non-equality into an equivalent equation. *Note that, if the problem that has an artificial variable and accordingly uses the big M method yields a final solution that has at least one artificial variable greater than zero, then the original problem has no feasible solution.*

### 3.6.9 Minimized Objective

Any minimization could be converted to an equivalent maximization by changing the signs of the variables of the objective function. By applying this to the objective function of Example 3.9 yields:

$$\begin{align*}
\text{Min} & \quad z = 0.4x_1 + 0.5x_2 \\
\text{Max} & \quad -z = -0.4x_1 - 0.5x_2
\end{align*}$$

In this case, the two formulations yield the same optimal solution as the smaller $Z$ is the larger (-$Z$) (i.e., the solution that gives the smallest value of $z$ in the entire feasible region must also give the largest value of (-$Z$) in this region.

_Solving Example 3.9_
First transform the problem by adding slack and surplus variables and changing the minimization to maximization:

\[
\begin{align*}
\text{Max} & \quad -z = -0.4x_1 - 0.5x_2 \\
\text{Subject to} & \quad 0.3x_1 + 0.1x_2 + s_1 = 2.7 \\
& \quad 0.5x_1 + 0.5x_2 = 6 \\
& \quad 0.6x_1 + 0.4x_2 - s_3 = 6 \\
& \quad x_1, x_2, s_1, s_3 \geq 0
\end{align*}
\]

Then, add artificial variables and subtract them times \( M \) from the objective function:

\[
\begin{align*}
\text{Max} & \quad -z + 0.4x_1 + 0.5x_2 + Ms_2 + Ms_4 = 0 \quad (0) \\
\text{Subject to} & \quad 0.3x_1 + 0.1x_2 + s_1 = 2.7 \quad (1) \\
& \quad 0.5x_1 + 0.5x_2 + s_2 = 6 \quad (2) \\
& \quad 0.6x_1 + 0.4x_2 - s_3 + s_4 = 6 \quad (3) \\
& \quad x_1, x_2, s_1, s_2, s_3, s_4 \geq 0
\end{align*}
\]

This problem has three basic variables \((s_1, s_2, s_4)\) for the initial basic feasible solution. The form presented in this set of equations is not proper to apply Gaussian elimination where each equation should have only one basic variable. Equation (0) has three basic variables \(z\) and \(s_2\) and \(s_4\). So, each basic variable should be eliminated from Eq. (0) except "\(z\)". To eliminate \(s_2\) and \(s_4\) from Eq. (0), subtract \(M\) times Eqs. (2) and (3) from Eq. (0), thus yields:

\[
-z + x_1 (0.4 - 1.1M) + x_2 (0.5 - 0.9M) + Ms_3 = -12M \quad (0)
\]

Then, the resulting initial simplex tableau for example 3.9 as follow (Table 3.16):

<table>
<thead>
<tr>
<th>Basic variables</th>
<th>Coefficient of Right-hand side</th>
</tr>
</thead>
<tbody>
<tr>
<td>(z)</td>
<td>-1 (-1.1M + 0.4) (-0.9M + 0.5) 0 0 M 0</td>
</tr>
<tr>
<td>(s_1)</td>
<td>0 0.3 0.1 1 0 0 0 0</td>
</tr>
<tr>
<td>(s_2)</td>
<td>0 0.5 0.5 0 1 0 0 0</td>
</tr>
<tr>
<td>(s_4)</td>
<td>0 0.6 0.4 0 0 -1 1 6</td>
</tr>
</tbody>
</table>

Then, applying the simplex method as usual as presented in Table 3.17.

<table>
<thead>
<tr>
<th>Table 3.17: Complete simplex tableau of Example 3.9</th>
<th>Table 3.17: Complete simplex tableau of Example 3.9</th>
<th>Table 3.17: Complete simplex tableau of Example 3.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>(z) (-1) (-1.1M + 0.4) (-0.9M + 0.5) (0) (0) (M) (0) (-12M)</td>
<td>(s_1) (0) (0.3) (0.1) (1) (0) (0) (0) (2.7)</td>
<td>(s_2) (0) (0.5) (0.5) (0) (1) (0) (0) (6)</td>
</tr>
<tr>
<td>(s_4) (0) (0.6) (0.4) (0) (0) (-1) (1) (6)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Table: Coefficient of Right-hand side

<table>
<thead>
<tr>
<th>Basic variables</th>
<th>$z$</th>
<th>$x_{1}$</th>
<th>$x_{2}$</th>
<th>$s_{1}$</th>
<th>$s_{2}$</th>
<th>$s_{3}$</th>
<th>$s_{4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z$</td>
<td>-1</td>
<td>-1.1M+0.4</td>
<td>-0.9M+0.5</td>
<td>0</td>
<td>0</td>
<td>M</td>
<td>0</td>
</tr>
<tr>
<td>$s_{1}$</td>
<td>0</td>
<td>0.3</td>
<td>0.1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$s_{2}$</td>
<td>0</td>
<td>0.5</td>
<td>0.5</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$s_{3}$</td>
<td>0</td>
<td>0.6</td>
<td>0.4</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
</tr>
</tbody>
</table>

| $z$             | -1  | 0       | 1/30(-16M+11) | 1/3(11M-4) | 0       | M       | 0       | -2.1M-3.6 |
| $x_{1}$         | 0   | 1       | 1/3       | 10/3      | 0       | 0       | 0       | 9      |
| $s_{2}$         | 0   | 0       | 1/3       | -5/3      | 1       | 0       | 0       | 1.5    |
| $s_{3}$         | 0   | 0       | 0.2      | -2       | 0       | -1      | 1       | 0.6    |

| $z$             | -1  | 0       | 0       | 1/3(-5M+7) | 0       | 1/3(-5M+5.5) | 1/3(8M-5.5) | -0.5M-4.7 |
| $x_{1}$         | 0   | 1       | 0       | 20/3      | 0       | 5/3      | -5/3      | 8      |
| $s_{2}$         | 0   | 0       | 0       | 5/3       | 1       | 5/3      | -5/3      | 0.5    |
| $x_{2}$         | 0   | 0       | 1       | -10      | 0       | -5       | 5         | 3      |

| $z$             | -1  | 0       | 0       | 0.5      | M-1.1   | 0       | M       | -5.25  |
| $x_{1}$         | 0   | 1       | 0       | 5        | -1      | 0       | 0       | 7.5    |
| $s_{3}$         | 0   | 0       | 0       | 1        | 0.6     | 1       | -1      | 0.3    |
| $x_{2}$         | 0   | 0       | 1       | -5       | 3       | 0       | 0       | 4.5    |

### 3.7 Sensitivity Analysis

As it has been discussed in Section 2.6, linear programming assumes certainty of the model parameters. However, these parameters are just estimates of quantities whose true values will not become known until the model is implemented at some time in the future. The main purpose of sensitivity analysis is to identify the sensitive parameters (those parameters that could not be changed without changing the optimal solution), to try to estimate these parameters more closely, and then to select a solution that remains a good one over the range of likely values of the sensitive parameters.

The sensitivity analysis measures how sensitive is the optimal solution to the change in the resources values (right hand side of the constraints), i.e., by changing the resource limits, would the optimal solution be changed and to what limit. This is called right hand side sensitivity analysis. Also, it measures the sensitivity of an optimal solution to changes in the values of the coefficients that multiply the decision variables in the objective function. This is called objective function sensitivity analysis.

#### 3.7.1 Right-Hand Side Sensitivity
Such type of sensitivity analysis is to answer questions about the validity of the model to the changes of the parameters and resources that the model assumes to be certain and constant. Consider the following model with its graphical representation (Fig. 3.4).

\[
\begin{align*}
\text{Max} & \quad z = 140x_1 + 160x_2 \\
\text{Subject to:} & \quad 2x_1 + 4x_2 \leq 28 \quad (1) \\
& \quad 5x_1 + 5x_2 \leq 50 \quad (2) \\
& \quad x_1 \leq 8 \quad (3) \\
& \quad x_2 \leq 6 \quad (4) \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

Consider the constraint \(2x_1 + 4x_2 \leq 28\), what would happen if the value of this constraint is more than 28 or less than 28? The right-hand side value determines the location of a constraint, while the coefficients of the variables determine the angle or orientation of that constraint, and the sense of the in-equality (\(\leq \) or \(\geq \)) determines which half-space created by that constraint is feasible. Accordingly, any change in the right-hand side value will change the position of the constraint up or down. When a constraint is moved in decision space, we must be concerned with how its movement affects the extreme points that define the possible locations for the optimal solution. The movement of a constraint could have no effect on the optimal solution, could change the value of the
optimal solution but not the extreme point, or could cause the optimal solution to move to a new extreme point.

If the right-hand side of constraint (2) is changed to 30, then this line will be shifted up from its current position (a) to a new position (b) as shown in Fig. 3.4, and the size of the feasible reign would increase by the hatched area. Accordingly, the new optimal solution would shift from point D to point D1 (Fig. 3.4).

3.7.2 **Objective Function Sensitivity**

The coefficients of the decision variables in the objective function could be based on uncertain data or subjective judgment of the decision maker. Accordingly, sensitivity analysis would be used to find the effect of changing these coefficients on the optimal solution. When we change the right-hand side value of a constraint, the constraint moved in decision space with a constant slope. However, when we change the coefficients of the objective function, the objective function slope will vary with the changing coefficient, but it will stay anchored at the optimal extreme point and rotate around it (Point D of Fig. 3.5). Consider the objective function of the same previous example:

$$\text{Max } z = 140x_1 + 160x_2$$

![Figure 3.5: Sensitivity to change in objective function](image)

Operations Research 80 Dr. Emad Elbeltagi
The optimal solution was found graphically by moving the objective function line through the feasible region until it last intersect the feasible region at one or more extreme point solutions. The coefficients of the objective function determine the slope of the objective function line, and thus determined which extreme point solutions are optimal. If we changed the coefficient of the $x_1$ of the objective function to 120, the objective function line will rotate counterclockwise; however, the optimal solution (point D2, Fig.3.5) will not change. Keeping decreasing this coefficient till it becomes 80, a new optimal solution will result (point D1, Fig. 3.5). Similarly, if we increased the coefficient of $x_1$ of the objective function till 160, the objective function line will rotate clockwise until it reaches a new optimal solution at point D1 (Fig. 3.5).

In simple problems, one can guess of the sensitive variables, however, when the problem size reached hundreds of variables and there is no graph to help in the explanation or understanding of the model results, the analyst must rely on the computer outputs to the relationships between problem data, variables values, and the optimal solution.

### 3.8 Solved Examples

#### 3.8.1 Example 1

Use the simplex method to determine the optimal solution for the following model:

Maximize $z = 3x_1 + 5x_2$

Subject to

1. $x_1 \leq 4$
2. $2x_2 \leq 12$
3. $3x_1 + 2x_2 \leq 18$

Solution

Standard form $z - 3x_1 - 5x_2 = 0$

$x_1 + s_1 = 4$

$2x_2 + s_2 = 12$

$x_1 + 2x_2 + s_3 = 18$

$x_1; x_2; x_3; s_1; s_2; s_3; and s_2 \geq 0$
Then, the optimum solution equals 36 and occurs at $x_1 = 2$; and $x_2 = 6$.

### 3.8.2 Example 2

The following tableaus were obtained in the course of solving linear programs with 2 nonnegative variables $x_1$ and $x_2$ and 2 constraints (the objective function $z$ is maximized). Slack variables $s_1$ and $s_2$ were added. In each case, indicate whether the linear program:

- is unbounded
- has a unique optimum solution
- has an alternate optimum solution
- is degenerate (in this case, indicate whether any of the above holds).

1. 

<table>
<thead>
<tr>
<th>Basic variables</th>
<th>Coefficient of</th>
<th>Right-hand side (solution)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$z$</td>
<td>$x_1$</td>
</tr>
<tr>
<td>$z$</td>
<td>1</td>
<td>-3</td>
</tr>
<tr>
<td>$s_1$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$s_3$</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

|                  | $z$  | 0     | 0     | 2.5   | 0     | 30    |
| $s_1$            | 0   | 1     | 0     | 1     | 0     | 0     | $4; 4/1 = 4$ |
| $x_2$            | 0   | 0     | 1     | 0     | $\frac{1}{2}$ | 0     | 6    |
| $s_3$            | 0   | 3     | 0     | 0     | -1    | 1     | $6; 6/3 = 2$ |

|                  | $z$  | 0     | 0     | 1.5   | 1     | 36    |
| $s_1$            | 0   | 0     | 0     | 1     | $\frac{1}{3}$ | -$\frac{1}{3}$ | 2  |
| $x_2$            | 0   | 0     | 1     | 0     | $\frac{1}{2}$ | 0     | 6    |
| $x_1$            | 0   | 1     | 0     | 0     | -$\frac{1}{3}$ | $\frac{1}{3}$ | 2  |

2.

<table>
<thead>
<tr>
<th></th>
<th>$z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z$</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z$</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>2</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-2</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>-2</td>
<td>0</td>
<td>3</td>
<td>6</td>
</tr>
</tbody>
</table>
3.

\[
\begin{array}{cccccc}
 z & x_1 & x_2 & s_1 & s_2 & \text{RHS} \\
1 & 2 & 0 & 0 & 1 & 8 \\
0 & 3 & 1 & 0 & -2 & 4 \\
0 & -2 & 0 & 1 & 1 & 0 \\
\end{array}
\]

4.

\[
\begin{array}{cccccc}
 z & x_1 & x_2 & s_1 & s_2 & \text{RHS} \\
1 & 0 & 0 & 2 & 0 & 5 \\
0 & 0 & -1 & 1 & 1 & 4 \\
0 & 1 & 1 & -1 & 0 & 4 \\
\end{array}
\]

Solution

a) The basic solution is optimal and unique since all variables in Row 0 have non-negative coefficients.

b) \(x_2\) has a negative coefficient in Row 0 but does not have a positive coefficient in other rows; hence the problem is unbounded.

c) Since \(s_1\), a basic variable, equals zero in the basic solution, the basic solution is degenerate but optimal.

d) The coefficient of \(x_2\), a non-basic variable is 0 in Row 0; hence the problem has an alternate optimal solution. Pivoting on Row 3 with \(x_2\) as the entering variable, we obtain an alternate solution.

\[
\begin{array}{cccccc}
 z & x_1 & x_2 & s_1 & s_2 & \text{RHS (solution)} \\
1 & 0 & 0 & 2 & 0 & 5 \text{ basic } x_2 = 4, s_2 = 8 \\
0 & 1 & 0 & 0 & 1 & 8 \text{ non-basic } x_1 = 0, s_1 = 0 \\
0 & 1 & 1 & -1 & 0 & 4 \text{ } z = 5 \\
\end{array}
\]

3.8.3 Example 3

A plant can manufacture five products \(p_1, p_2, p_3, p_4\) and \(p_5\). The plant consists of two work areas: the job shop area \(A_1\) and the assembly area \(A_2\). The time required to process one unit of product \(p_j\) in work area \(A_i\) is \(t_{ij}\) (in hours), for \(i = 1, 2\) and \(j = 1, \ldots, 5\). The weekly capacity of work area \(A_i\) is \(C_i\) (in hours). The company can sell all its products \(p_j\) at a profit of \(s_j\), for \(j = 1, \ldots, 5\). The plant manager thought of
writing a Linear Program (LP) to maximize profits, but never actually did for the following reason: From past experience, he/she observed that the plant operates best when at most two products are manufactured at a time. He/she believes that if he/she uses linear programming, the optimal solution will consist of producing all five products. Formulate a LP model. Do you agree with the plant manager? Explain, based on your knowledge of linear programming.

Solution

This LP model consists of an objective function to maximize the profit and two constraints represented the available weekly production hours at each area. The variables are how many units of each $p_1$, $p_2$, $p_3$, $p_4$ and $p_5$ to produce.

$$\text{Max} \quad p_1s_1 + p_2s_2 + p_3s_3 + p_4s_4 + p_5s_5$$

$$\text{Subject to} \quad (p_1 + p_2 + p_3 + p_4 + p_5) t_1 \leq C_1$$
$$\quad \quad \quad \quad \quad (p_1 + p_2 + p_3 + p_4 + p_5) t_2 \leq C_2; \text{ and}$$
$$\quad \quad \quad \quad \quad p_1, p_2, p_3, p_4 \text{ and } p_5 \geq 0$$

A LP model always has an optimal basic solution as long as it is feasible and bounded. In such solutions, there will always be a fixed number of basic variables which can be greater than zero; the rest will be non-basic and hence zero. The number of basic variables will equal the number of constraints. In this example, there are only two constraints, so two variables will always be basic and hence at most two variables may be greater than zero. Hence, the optimum solution will occur only when at most two products are produced at a time.

3.8.4 Example 4

Use the simplex method to determine the optimal solution for the following model:

$$\text{Maximize} \quad z = 4x_1 + 6x_2$$

$$\text{Subject to} \quad x_1 + x_2 \geq -4 \quad (1)$$
$$\quad \quad 3x_1 - 2x_2 \leq 6 \quad (2)$$
$$\quad \quad x_1 + x_2 \geq 5 \quad (3)$$
$$\quad \quad x_1 + x_2 \leq 10 \quad (4)$$
$$\quad \quad x_1, \text{ and } x_2 \geq 0$$
Solution

Augmented form  
\[ z - 4x_1 - 6x_2 + M s_4 = 0 \]  
\[ -x_1 - x_2 + s_1 = 4 \]  
\[ 3x_1 - 2x_2 + s_2 = 6 \]  
\[ x_1 + x_2 - s_3 + s_4 = 5 \]  
\[ x_1 + x_2 + s_5 = 10 \]

Then, subtract M times Eq.(4) from Eq. (0), thus yields:

\[ z - x_1(M+4) - x_2(M+6) + M s_3 = -5M \]

<table>
<thead>
<tr>
<th>Basic variables</th>
<th>Coefficient of</th>
<th>Right-hand side</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z )</td>
<td>-M-4</td>
<td>0</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>-M-6</td>
<td>0</td>
</tr>
<tr>
<td>( s_1 )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>( s_3 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( s_4 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( s_5 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( z )</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( s_1 )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( s_3 )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( s_4 )</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( s_5 )</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

3.9 Exercises

1. Transform the following linear program model into standard form for the simplex method.

Minimize  
\[-2x_1 + 3x_2\]  
\[x_1 - 3x_2 + 2x_3 \leq 3\]  
\[-x_1 + 2x_2 \geq 2\]  
\[2x_2 \leq -5\]  
\[x_1 \geq 0; \ x_2 \geq 0; \ and \ x_3 \text{ unrestricted}\]

2. Solve analytically and verify graphically, the following LP model:
Maximize \[ z = x_1 + x_2 \]
Subject to \[
\begin{align*}
& x_1 - 2x_2 \leq 4 \\
& 2x_1 + 3x_2 \leq 12 \\
& 3x_1 + 4x_2 \leq 12 \\
& x_1 \geq 0; \ x_2 \geq 0
\end{align*}
\]

3. Use the simplex method to solve the problem:

Maximize \[ z = 3x_1 + 2x_2 \]
Subject to \[
\begin{align*}
& x_1 + 2x_2 \leq 6 \\
& 2x_1 + x_2 \leq 8 \\
& -x_1 + x_2 \leq 1 \\
& x_2 \leq 2 \\
& x_1 \geq 0; \ x_2 \geq 0
\end{align*}
\]

a. Determine the basic feasible solution at each iteration.
b. Define the basic and non-basic variables at the optimal solution.
c. Determine the maximum value of \( z \).

4. Use the simplex method to solve the problem:

Maximize \[ z = x_1 + 2x_2 + 3x_3 \]
Subject to \[
\begin{align*}
& 2x_1 + x_2 + x_3 \leq 4 \\
& x_1 + 2x_2 + x_3 \leq 4 \\
& x_1 + 2x_2 + 2x_3 \leq 4 \\
& x_1 + x_2 + x_3 \leq 3 \\
& x_1 \geq 0; \ x_2 \geq 0; \ x_3 \geq 0
\end{align*}
\]

a. Determine the basic feasible solution at each iteration.
b. Define the basic and non-basic variables at the optimal solution.
c. Determine the maximum value of \( z \).

5. Use the simplex method to solve the problem:

Maximize \[ z = x_1 + 2x_2 + 4x_3 \]
Subject to \[
\begin{align*}
& 3x_1 + x_2 + 5x_3 \leq 10
\end{align*}
\]
\begin{align*}
  x_1 + 4x_2 + x_3 & \leq 8 \\
  2x_1 + 2x_3 & \leq 7 \\
  x_1 + x_2 + x_3 & \leq 3 \\
  x_1 \geq 0; \ x_2 \geq 0; \ x_3 \geq 0
\end{align*}

a. Determine the basic feasible solution at each iteration.

b. Define the basic and non-basic variables at the optimal solution.

c. Determine the maximum value of \( z \).

6. Consider the following linear program and the associated simplex tableau:

Maximize \( 5x_1 + 3x_2 + x_3 \)

Subject to \( x_1 + x_2 + x_3 \leq 6 \)

\( 5x_1 + 3x_2 + 6x_3 \leq 15 \)

\( x_1; x_2; x_3 \geq 0 \)

<table>
<thead>
<tr>
<th>( z )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td>15</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0.4</td>
<td>-0.2</td>
<td>1</td>
<td>-0.2</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0.6</td>
<td>1.2</td>
<td>0</td>
<td>0.2</td>
<td>3</td>
</tr>
</tbody>
</table>

(a) What solution does this tableau represent? Is it optimal? Why or why not?

(b) Does this tableau represent a unique optimum? If not, find an alternative optimal solution.

7. Suppose the following tableau was obtained in the course of solving a linear program with nonnegative variables \( x_1, x_2, x_3 \) and two inequalities. The objective function is maximized and slack variables \( s_1 \) and \( s_2 \) were added.

<table>
<thead>
<tr>
<th>( z )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>a</td>
<td>b</td>
<td>0</td>
<td>4</td>
<td>82</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-2</td>
<td>b</td>
<td>0</td>
<td>3</td>
<td>c</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>3</td>
<td>0</td>
<td>-5</td>
<td>3</td>
</tr>
</tbody>
</table>

Give conditions on \( a, b \) and \( c \) that are required for the following statements to be true:
(i) The current basic solution is a feasible basic solution. Assume that the condition found in (i) holds in the rest of the exercise.

(ii) The current basic solution is optimal.

(iii) The linear program is unbounded (for this question, assume that \( b > 0 \)).

(iv) The current basic solution is optimal and there are alternate optimal solutions (for this question, assume \( a > 0 \)).

8. Use the simplex method to solve the problem:

Maximize \( z = x_1 + 2x_2 + 2x_3 \)
Subject to
\[
5x_1 + 2x_2 + 3x_3 \leq 15 \\
x_1 + 4x_2 + 2x_3 \leq 12 \\
2x_1 + x_3 \leq 8 \\
x_1 \geq 0; \ x_2 \geq 0; \ x_3 \geq 0
\]

a. Determine the basic feasible solution at each iteration.

b. Define the basic and non-basic variables at the optimal solution.

c. Determine the maximum value of \( z \).

9. Solve the following linear program using the simplex method. Compute the value of the objective function and decision variables at optimality, and indicate which statement best describes the solution and why: 

a) this linear program has a unique optimal solution; 
b) this linear program has alternate optima; 
c) this linear program is infeasible; or 
d) this linear program is unbounded.

Maximize \( z = 3x_1 + 2x_2 \)
Subject to
\[
-x_1 + 2x_2 \leq 6 \\
2x_1 - 5x_2 \leq 10 \\
2x_1 + 2x_2 \leq 2 \\
x_1 \geq 0; \ x_2 \geq 0
\]

10. Solve the following linear program using the simplex method. Compute the values of the objective function and decision variables at optimality, and indicate which statement best describes the solution and why: 

a) this linear program has a
unique optimal solution; b) this linear program has alternate optima; c) this linear program is infeasible; or d) this linear program is unbounded. Verify your solution graphically.

Minimize \[ z = 4x_1 - 5x_2 \]
Subject to
\[-6x_1 + 3x_2 \leq 12 \]
\[4x_1 - 2x_2 \leq 24 \]
\[3x_1 + 2x_2 \leq 30 \]
\[x_2 \leq 6 \]
\[x_1 \geq 0; \ x_2 \geq 0 \]

11. Solve the following linear program using the simplex method. Compute the value of the objective function and decision variables at optimality, and indicate which statement best describes the solution and why: a) this linear program has a unique optimal solution; b) this linear program has alternate optima; c) this linear program is infeasible; or d) this linear program is unbounded.

Minimize \[ z = 2x_1 + 3x_2 + 2x_3 \]
Subject to
\[2x_1 + x_2 + x_3 \leq 4 \]
\[x_1 + 2x_2 + x_3 \leq 7 \]
\[x_1 + 2x_2 + x_3 \leq 12 \]
\[x_1; \ x_2; \ x_3 \geq 0 \]

12. Solve the following linear program using the simplex method. Compute the value of the objective function and decision variables at optimality, and indicate which statement best describes the solution and why: a) this linear program has a unique optimal solution; b) this linear program has alternate optima; c) this linear program is infeasible; or d) this linear program is unbounded.

Minimize \[ z = 2x_1 + 3x_2 + x_3 \]
Subject to
\[2x_1 + x_2 - x_3 \geq 3 \]
\[x_1 + x_2 + x_3 \geq 2 \]
\[x_1; \ x_2; \ x_3 \geq 0 \]
13. Consider the simply supported beam shown below. The maximum reactions that supports “a” and “b” can carry are 12 and 19 tons, respectively. The beam can withstand a maximum bending moment of 65 t.m. that may occur at one of the load points (c or d).

![Beam Diagram]

Formulate a linear program model that will determine the maximum possible loads \( P_1 \) and \( P_2 \) that this system can carry. Solve your model using the simplex method.

14. A company manufactures three different types of pipe fittings: tees, elbows, and splicers. Daily production of these parts are limited by the availability of lathe time, grinder time and labor availability as shown below.

<table>
<thead>
<tr>
<th>Resources</th>
<th>Products</th>
<th>Availability of resources</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>100 tees</td>
<td>100 elbows</td>
</tr>
<tr>
<td>Person hours</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>Lathe hours</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Grinder hours</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Profit per 100 units (LE)</td>
<td>700</td>
<td>550</td>
</tr>
</tbody>
</table>

a. Formulate a linear program that will suggest a production policy for maximizing daily profit.

b. Set the augmented form by adding the appropriate slack and surplus variables.

c. Solve for the optimal solution using the simplex method.

d. What is constraining the present production level?