LECTURE NOTES
ON
OPERATIONS RESEARCH

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PREFACE

In the Name of ALLAH the Most Merciful, the Most Compassionate

All praise is due to ALLAH and blessings and peace be upon His messenger and servant, Muhammad, and upon his family and companions and whoever follows his guidance until the Day of Resurrection.

Operations research is a relatively young field, having emerged during the middle decades of the twentieth century. However, its impact has been quite remarkable. It has become a standard tool for improving the efficiency of business operations around the world. This book deals with some tools of a larger field called operations research.

This book is dedicated mainly to undergraduate engineering students, especially Civil Engineering students where most of the applications are presented in the civil engineering field. It provides the reader with the main knowledge to model a given system using linear programming and solving such linear model. It includes five chapters: Chapter 1 provides a general introduction to types of models and the main steps of modeling a system. Chapter 2 introduces the principles of mathematical modeling using linear programming and the graphical solution to solve the model. The simplex method for solving linear programming models is presented in chapter 3. Chapter 4 is dedicated for modeling and solving the transportation and assignment problems. Finally, chapter 5 is dealing with the decision analysis techniques. Many solved examples have been added to enable the students to understand the material presented in this book. Also, each chapter is followed by exercises for training purposes.

Finally, May ALLAH accepts this humble work and I hope it will be beneficial to its readers.
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CHAPTER 1

INTRODUCTION

1.1. The Origin of the Operations Research Science

After the industrial revolution and change in organization size and complexity, it becomes more and more difficult to allocate and utilize resources efficiently. However, this science became more advanced since the Second World War as there was an urgent need to allocate the scarce resources to the various military operations. Another factor that added greatly to this science is the computer revolution.

As it is seen from the name of the course “system analysis” or “analysis of the systems”, so it is related to how to conduct and coordinate operations within a system or organization. It can be applied to many areas such as, financial planning, health care, military, public services, construction, etc.

System analysis “operation research” frequently attempts to find a best solution – called the optimum solution - for a given problem.

1.2. Overview of System Analysis Modeling Approach

1. Define the problem and gather relevant data

Mostly, when a practical problem described, it is described in a vague and imprecise way. Therefore, the first order is to develop a well-defined statement of the problem, i.e., determining such things as the appropriate objectives, constraints, inter-relationships, possible alternative courses of action, time limits for making a decision, etc. It is difficult to extract a right answer from the wrong defined problem.
It is necessary to seek solutions that are optimal for all variables (the whole organization) rather than suboptimal solutions that are best for only one component. The problem of privatizing the water network of a city generally affects four parties: the owner who desires profit; the employees who desire steady employment; the people who desire low priced and high quality water; the government which desire a continuous services and fair taxes.

2. **Formulate a mathematical model to represent the problem**

After defining the problem, the next phase is to reformulate this problem in a form that is suitable for analysis. **Models or idealized representation** can be used to express any daily life action such as the motion of a car, chemical reactions, etc. For example, the famous law $F = ma$, is a sample of mathematical model. So, mathematical model is the system of equations that describe the problem. For example, if there are some factors that affect the profitability of an organization (such as number of products, price, production time, etc.) and they need to be determined. These factors are called the **decision variables** and can be denoted mathematically as $x_1, x_2, ..., x_n$. If the profit is measured in terms of these variables as $p = ax_1 + bx_2 + .... + cx_n$, this is called the **objective function**. Any restrictions that may be applied to these variables can be also represented mathematically and called **constraints**. These constraints can be represented in terms of inequalities or equations such as, $x_1 \geq 0$, and $dx_2 - x_3 \leq k$, etc.

The mathematical model, in this case is to choose the values for the decision variables so as to maximize the objective function and respect the given constraints. One of the most used mathematical models is the **linear programming models**, where the mathematical functions appear in both the objective function and the constraints are all linear functions.
Mathematical models are very essential and have many advantages over verbal description of the problem, as they describe the problem more precisely and accordingly make the problem more comprehensible. It allows the use of computers to analyze the problem.

Sometimes, it is necessary to do some simplifying assumptions to make the problem solvable. Therefore, care must be taken to ensure that the model remains a valid representation of the problem.

*For instance, consider the problem of the construction site layout planning. This problem is described verbally as: it is required to layout the temporary facilities on a construction site so as to increase safety, visibility, and to reduce traveling time within the site.*

3. **Develop a procedure for driving solutions to the problem**

   The next step after mathematical formulation for a given problem is to develop a procedure for solving the problem. For instance, this procedure may be a computer program. Sometimes, software shells might be used for solving mathematical models. It is necessary, that the procedure being able to search for an optimal, or best, solution.

   Sometimes, obtaining an optimal solution for a problem becomes very difficult and time consuming and may be even impossible. In such case, *heuristic procedures* may be used. Heuristic procedures are rule-based methods that seek good or sub-optimal solutions.

4. **Test the model and refine it as needed**

   The procedure used to solve the model need to be tested to make sure that the procedure is error-free. The model can be tested against benchmark problems in the same area that is developed for. This process of testing the model is usually called *model validation.*
5. **Prepare for the application of the model**

   After the testing of the model has been completed and an acceptable model has been developed, the next step is to install a well-documented system for applying the model. These documents include the model, solution procedure, and operating procedures for implementation.

6. **Implementation**

   The last step is to implement the system. This is the most important step because it ensures that the model is translated into an operating procedure.

1.3. **Types of Models**

1.3.1 **Descriptive versus Prescriptive**

   Models are always built to assist in the design or management of natural or constructed process. Many people refer to these models as Mathematical Models. For the most part, models from various disciplines were built using differential and difference equations to explain, to comprehend, and to predict natural phenomena. These mathematical representations are called *descriptive models* because they offer, for a given set of inputs and initial conditions, a description of the outputs through time the phenomena under study. Descriptive models is said to answer the question, “*If I follow this course of action, what will happen?*” The descriptive model predicts the quantitative outcomes possibly through time.

   Prescriptive models, on the other hand, focuses on the science of decision making and policy development. The invention of a mathematics of decision making along with the continual improvement of digital computers make it possible for the development of more powerful and complicated models. The representation that uses the mathematics of decision making is called a prescriptive model because it prescribes a course of action, a design, or a policy. Prescriptive models is said to answer the question, “*What is the best course of action that I might follow?*” The
prescriptive model finds and suggests the best strategy to choose from all possible strategies. Another term used for a prescriptive model is to call it an optimization model. The material presented in these notes will focus on prescriptive models.

1.3.2 Deterministic versus Stochastic

In deterministic models, parameter values are determined and known at the outset. Models of this type are characterized by their data elements are not thought of as variable but as relatively fixed and predictable quantities. For example, given the initial contents of the stockpile and a specified release of materials and a stated purchase or manufacture of new materials during a unit of time, a deterministic model suggests that there is just one possibility for the final end-of-period condition of the stockpile. That is, only a single outcome can occur from a month’s events given a specific choice of action.

In contrast to deterministic models, other models might utilize data elements that are not precisely known but can be characterized by a mean and some random variation about the mean. For example, predicting the flow in a reservoir during a specific month may not be accurately defined as in some years the flow may be high and on other years it may be low. Models in which the data elements are random or variable (i.e., capable of taking on any value from a range of values) are called stochastic models. Given an initial value of the storage in the reservoir, and a known amount of release for water supply, the stochastic model suggests that the end-of-month contents of the reservoir can be stated, but with some uncertainty. This is because of the random inflow to the reservoir.

1.3.3 Statistical Models

Positioned in concept somewhere between the deterministic and stochastic models is a statistical model. In a statistical model, system inputs have been observed, and system outputs have been measured. The relationship between systems inputs and outputs does not seem to be consistent. A statistical model is a hypothesis of the
relationship between output and input. The model may suggest that the relationship is linear or nonlinear. A statistical model may be thought of as neither deterministic nor stochastic, but a model that provides the most likely or expected outcome of conditions given the input and uncertain events.

1.4. Rules for Modeling

There is no formula exists for the way in which models should be built. However, there are approaches that can generally work. The following are some of the rules that might be followed:

- Keep the model as simple as possible while still answering the questions at hand. Try to use parameters - such as size, temperature, elevation, cost, distance, and so on – whose values can be obtained or estimated reasonably. The level of uncertainty about the values themselves should be relatively small.

- Let \( x \) equal the unknown level of a decision or, where many decisions are involved, let \( x_j \) equal the unknown level of the \( j^{th} \) decision.

- Try to list all possible constraints and to formulate all the objective or objectives. A constraint is what you must achieve without fail. An objective is a goal you would like to achieve as nearly as possible. The relative achievement of a goal is the way you evaluate the merit of alternative solutions.

- Final step is solving the model. This would be probably the easiest thing. When processes are additive and proportional, the model by this mean is called linear.

1.5. Sample Modeling

1.5.1 Sample Model 1

This model deals with building a reservoir to supply water to a city. In order to model this problem, we need to define data requirements, objectives or goals, decision variables, and the constraints defining acceptable performance. In terms of
data, we need the projected volumetric demand for water in each month for the future year for which the system is being planned. We need also to know, by month, the worst historical sequence of volumetric inflows to the reservoir. This should be obtained from long records of historical stream flows. Our goal might be the smallest capacity reservoir necessary to provide the projected demand throughout the duration of the worst flow ever recorded. The decision variables would include not only the reservoir capacity but also the storages planned for the end of each month, given that the required monthly water supplies will always be capable of delivery. Two kinds of constraints may be formulated. First, no more water can be stored in the reservoir than the reservoir capacity level. Water that arrives in excess of this capacity after delivery of the monthly water demand must be released to the stream. Second, in order to be capable always of delivering the needed supply, the reservoir must never go dry before a month is up. To achieve that condition, end-of-month reservoir contents must always be greater than or equal to zero.

1.5.2 Sample Model 2

This model deals with the development and expansion of an electric power system for a specific region. Let us assume that the new power station will be sited not far from the grid network of existing power lines. Demand is expected to grow over the next twenty years, so new power station is needed to supply that demand. In addition to demand, other needed data include the cost to build and operate various sizes of hydroelectric plants, cool-fired electric plants, and nuclear power plants. Proportional power losses along the segments of the network would likely be important information to have as well. The objective is to meet demands for power at the least total cost where cost is the cost of building and operating the expanded system of power plants. The constraints are that each city must be assigned sufficient power resources from among all the plants, previously established or newly built. Decision variables may be considered building a plant of specific type or not. For example, decision variables may be 1 or 0. A variable for a plant of type \( k \) at site \( i \) built to size \( j \) would be 1 if such plant were established and 0 other wise.
1.6. Exercises

1. A construction contractor employs a given force of skilled workers and has a limited set of specialized machinery. There a number of categories of skilled workers, as well as a number of classes of specialized machinery. The contractor has a number of construction projects underway and has committed the company to completion dates for each of them. Certain activities are common to the various projects and require the same inputs of labor and equipment. The activities occur on various dates throughout the time span from project start to project completion. However, the contractor does not have enough labor and equipment to work on these activities on the same day on all projects. Hence, the contractor wants to schedule the specialized work forces and specialized equipment on the projects so that ideally no labor or equipment shortages do exist, extra costs must be incurred for temporary workers or for equipment. A predetermined payment has already been received for each project. Now, the contractor wants to keep construction costs to its minimum. Define:
   a. Decision variables;
   b. Parameters;
   c. The objective function in words; and
   d. Constraints in words.

2. The design engineer for a large building must determine the foundation type and size that will ensure that excessive settlement of the building does not occur. Damage costs to the building increase with the degree of settlement. The engineer has some idea of the probability of finding certain soil strength, but a better estimate can be obtained from field experiments and from laboratory tests. There are many possible experiments are available, and generally the reliability of an experimental test result increases with the cost of the test. However, for every test there is still a chance that the test result is not representative of the average soil condition. From past experience and data, the engineer can estimate the probability of having a certain soil given a particular experimental result from a
particular test. The engineer wants the total expected cost of designing, construction, and maintaining the building to be a minimum including the costs of soil testing. Define:

a. Decision variables;
b. Parameters;
c. The objective function in words; and
d. Constraints in words.

3. A contractor can sell several classes of concrete at a different price per cubic meter for each class. The materials specification for each class of concrete allows the percentage by weight of cement, sand, and gravel to range between certain upper and lower bounds. The contractor knows the unit cost to the company for each of the three components of concrete, and how much of each component that is available for the company to purchase. The amount of concrete of each type that the contractor can sell is limited but known. (Hint: consider a maximization objective function, not minimization). Define:

a. Decision variables;
b. Parameters;
c. The objective function in words; and
d. Constraints in words.

4. A single reinforced rectangular concrete beam must carry a known imposed moment and shear. The span length is also known, and the deflection of the beam must not exceed a certain value. The width and depth of the beam are to be determined, as well as the area of the steel to be placed in the bottom of the beam. The cost of concrete per cubic meter and the cost of steel per kilogram are known, as well as the compressive strength of the concrete and yield strength of the steel. The designer wants to design the least-cost beam. The code for such beams states that a certain minimum amount of steel, as a percent of the total effective cross sectional area, must be present in order to avoid excessive cracking on the bottom of the beam due to temperature fluctuations. The code also gives a limit on the
maximum amount of steel, again expressed as a percent of the total effective cross sectional area of the beam, which can be present to avoid sudden compressive failure in the concrete at the top of the beam. Define:

a. Decision variables;
b. Parameters;
c. The objective function in words; and
d. Constraints in words.

5. An agency is planning a new toll exit for an existing toll highway. The number of toll booths to put at the exit is in question. The agency wants to keep costs low by having as few booths as possible. But if the waiting lines get too long during rush hours and other peak periods will hurt public comfort, reduce the number of people who will use the exit, and in the worst case, back waiting vehicles onto the highway may cause hazardous situation. The agency believes that no more than six cars, on average across the lane, should be stored in the waiting lines during rush hours, but is willing to examine other average waiting line lengths. From data elsewhere, it estimates the arrivals at the exit during each two-minute segment of the rush hour. It knows that it takes 17 seconds to service a car at a booth, resulting in 3.53 vehicles being processed each minute by a toll attendant. The agency decides to develop an optimization model to analyze the problem. The goal is to assess the impact of the number of toll booths on the average length of the waiting line. Define:

a. Decision variables;
b. Parameters;
c. The objective function in words; and
d. Constraints in words.

6. A city needs a certain amount of water from a river that is 100 m below and 5 kilometers from the city’s water treatment plant. The city engineer must decide on the internal diameter of the pipeline to the city and the head capacity of the pump to be installed at the river. The engineer knows that a smaller pipe is less
expensive but results in greater head loss due to friction and therefore requires a greater head capacity for the pump, which increases the cost of the pump. Conversely, a larger pipe is more expensive, but permits a less expensive pump to meet the fixed head and flow requirement at the water treatment plant. The engineer is also aware that the range of pump capacities and pipeline diameter is limited in order to avoid a long waiting period for delivery. The most economical pump/pipe combination is desired. Define:

a. Decision variables;
b. Parameters;
c. The objective function in words; and
d. Constraints in words.